PURITY AND ITS ALLIED CONCEPTS
IN
SOME SPECIAL MODULES

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CERTIFICATE

This is to certify that the contents of this thesis entitled " Purity and its allied concepts in some special modules " is an original work of Mr. Abdul Halim Ansari, done under my supervision.

I, further, certify that the work of this thesis, either partially or fully has not been submitted to any other institution for the award of any other degree.

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PREFACE

It is well known that pure subgroups, neat subgroups, basic subgroups, high subgroups, large subgroups and divisible groups etc. have become most useful tools in abelian groups. The problem of generalising these concepts and their properties for various types of modules have been studied from time to time. Most of the results in abelian groups have been generalised for unital modules over principal ideal domains with identity. It is, however, a very delicate question, to consider the class of rings that, for the modules over these rings, the results of abelian groups can be carried. Hence these studies were carried by imposing some restrictions either on modules or the rings involved. The latter type of study has been more often done. For instance, in 1952, I. Kaplansky generalised some of the well known results of pure subgroups and divisible groups for modules over Dedekind rings and valuation rings. He proved a very fundamental result : 'Any finitely generated module over an almost maximal valuation ring is direct sum of cyclic modules'. Latter, in 1970, D. Eisenbud and J. C. Robson studied modules over

Dedekind prime rings. Subsequently, D. Eisenbud and P. Griffith studied serial rings and modules over these rings and did some very useful decomposition theorems. In 1972, H. Marubayashi, generalised some of the results of torsion abelian groups for torsion modules over bounded Dedekind prime rings. Latter, in 1975, S. Singh [33] did the analogous study of modules over bounded hereditary Noetherian prime rings and generalised some of the results of abelian groups. In [34] he further introduced the concept of h-purity and generalised some of his own results of [33]. Recently, in 1977, M. Zubair Khan [28] studied the module satisfying two conditions as introduced by Singh [34] and called the module as S_2-module. Analogously, M. Zubair Khan also introduced many new concepts like h-neat submodules, complement submodules, high submodules, h-divisible modules, basic submodules etc. for S_2-modules in [20, 22, 23, 24, 25, 26, 28] and to a great extent generalised a number of well known results of torsion abelian groups. Anyhow, his contribution is very fundamental and is different from the others in the sense that his work does not depend upon the nature of the rings involved and the results of abelian groups have been nicely carried over.

Our main purpose of the present thesis is to generalise some more fundamental concepts and results of torsion abelian groups which were still untouched. Therefore, on account of their importance the need was felt to introduce them for S_2-modules.

The present thesis comprises of five chapters, consisting of various sections, the sections being numbered in the order in which they occur in the thesis. Most of the results of this thesis are either communicated or yet to appear in Tamkang J. Math. [1], Tamkang J. Math. [2] and Proc. Amer. Math. Soc. [30].

The principal purpose of the introductory chapter I on preliminaries, is to acquaint the readers with the terminology and basic results of modules, which will be more often used in the subsequent chapters. This chapter is also intended to make the thesis as much self contained as possible. Here we have given some definitions and properties of h-pure, h-neat submodules, h-divisible modules and basic submodules and no originality is claimed.

In chapter II, h-pure-complete modules and complement submodules in S_2-modules have been dealt with and a necessary and sufficient condition for an element of an S_2-module to be embeddable in a uniform summand of finite length has been obtained (Theorem 4.10). Further it is proved that if M is an S_2-module with elements of infinite height and of finite height, then M possesses a non-trivial decomposition (Theorem 4.11). It is also proved that if M and K are S_2-modules then $M \oplus K$ is h-*pure*-complete under some conditions (Theorem 5.7). Finally, we have obtained a very interesting theorem which is a characterisation of S_2-modules in which every submodule is h-neat (Theorem 6.4).

In chapter III, we have introduced some more new concepts like h-dense submodule, h-dense subsocle and large submodules and proved that an h-neat submodule of an S_3-module supported by an h-dense subsocle is h-pure (Theorem 7.9). It is also proved that if M is an S_2-module with some conditions and if some large submodule of M is decomposable then every large submodule of M is decomposable (Theorem 8.7). Lastly, we have shown that a high submodule of a large submodule is closed in a high submodule of the module itself (Theorem 9.6).

In chapter IV, the concept of fair module has been introduced and we constructed a very useful example which shows that every module is not fair (Example 10.2). We have shown that if an S_2-module M is either h-divisible or is a direct sum of uniserial modules of length n and $n+1$, then M is a fair module (Theorem 11.4). It is further proved that a reduced fair S_3-module is bounded (Proposition 12.3). Lastly, we have obtained a characterisation of fair module and proved that if M is an S_3-module then M is fair module if and only if M is either h-divisible or is a direct sum of uniserial modules of length n and $n+1$ for some n (Theorem 12.5).

In the last chapter, we have discussed the submodules $H^k(N)$ as introduced by M. Zubair Khan in [20] and proved a number of basic results. For instance, it is proved that for any two submodules N and K of M, $H_m(N \cap H^{m+n}(K)) = H_m(N) \quad H^n(K)$ (Proposition 13.4) and $H_n(H^n(N)) = N \cap H_n(M)$. We have also deduced some useful results on horizontal exponent. As an application it has been shown that a submodule N of an S_2-module M is h-pure if and only if $H_n(H^n(N)) = H_n(N)$ for all $n \geq o$ (Proposition 15.1). It is further proved that for any submodule

N of M the following are equivalent :

(a) N is h-pure in M,

(b) N is a direct summand of $H^n(N)$ for every $n \geq o$,

(c) If $N \subseteq K \subseteq M$ and $K/_N$ is finitely generated then N is a direct summand of K (Theorem 15.4).

CHAPTER - I

PRELIMINARIES

The concepts of pure subgroups, neat subgroups, divisible subgroups, basic subgroups and high subgroups are quite important objects in abelian groups. Most of these concepts have been generalised by M. Zubair Khan [20, 21, 22, 23, 24, 25, 26, 27, 28, 29] and S. Singh [34] for a special type of module. The principal purpose of this introductory chapter is to recall some necessary definitions, notations and other background informations needed for the subsequent chapters. This is being done only to fix up the terminology and notations for subsequent use, and no originality is claimed. In section 1, some definitions and elementary properties of modules are given. The elementary concepts and properties of S_2-modules and S_3-modules introduced in [20] and [2] respectively have also been given. In section 2, we have given some very useful definitions and results on h-pure, h-neat and high submodules as done in [21, 23, 24, 28, 34]. In section 3, we have recalled some of the results of h-divisible and basic submodules from [25,26].

Throughout we shall consider right R-module M_R where R is an associative ring with identity.

§ 1. Some elementary concepts.

Definition (1.1). A module M_R is called simple if M has no proper submodules.

Definition (1.2). Let M_R be a module then the sum of all simple submodules of M is called socle of M and is denoted by Soc(M).

It is easy to see that for any submodule K of M_R, $Soc(K) = K \cap Soc(M)$ and $Soc(Soc(M)) = Soc(M)$.

Proposition (1.3) [5,p.121]. If $\{M_\alpha\}_{\alpha \in \Delta}$ is an indexed set of submodules of M with $M = \oplus \sum_{\alpha \in \Delta} M_\alpha$ then $Soc(M) = \oplus \sum_{\alpha \in \Delta} Soc(M_\alpha)$.

Definition (1.4). Let N be a submodule of M_R then N is called essential submodule of M if $N \cap T \neq 0$ for every non-zero submodule T of M. Also M is called an essential extension of N.

<u>Proposition</u> (1.5). If N is an essential submodule of M then $Soc(N) = Soc(M)$.

<u>Definition</u> (1.6). If N and K are submodule of a module M then N is called complement of K if N is maximal with respect to the property $N \cap K = o$. A submodule T of M is called complement submodule if T is a complement of some submodule U of M.

<u>Proposition</u> (1.7) [3,p.15]. If N is a submodule of M and K is any complement of N in M, then there exists a complement Q of K in M such that $N \subseteq Q$. Furthermore, any such Q is a maximal essential extension of N in M.

<u>Definition</u> (1.8). A submodule N of M is called a direct summand of M if there exists a submodule K of M such that $M = N \oplus K$. K is called the complementary summand of M.

<u>Definition</u> (1.9). A submodule N of M is called absolute direct summand of M if for every complement K of N in M, $M = N \oplus K$.

Definition (1.10). A module M_R is called uniform if intersection of any two of its nonzero submodules is nonzero.

Definition (1.11). Let M be a nonzero module. Then a finite chain of submodules of M,

$$M = M_o > M_1 > M_2 > \dots > M_n = o$$

is called a composition series of length n for M provided M_{i-1}/M_i is simple $(i = 1, 2, \dots, n)$. If M is a module and its length is n then we write $d(M) = n$.

Definition (1.12). A module M_R is called uniserial if it has a unique composition series of finite length.

From the definition it follows that uniserial modules are totally ordered.

Definition (1.13). Let N be a submodule of M_R then $\{ r \in R \mid xr = o \text{ for every } x \in N \}$ is called annihilator of N and is denoted by $\operatorname{ann}(N)$.

Definition (1.14). A module M_R is called divisible if $Mc = M$ for all regular elements $c \in R$.

Definition (1.15). A module M_R is called projective if given any diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow g & & \\ A & \xrightarrow{f} & B & \longrightarrow & 0 \end{array}$$

of R-modules with exact row, it is always possible to find an R-homomorphism $h : M \longrightarrow A$ such that $fh = g$.

Definition (1.16). A module M_R is called injective if given any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow g & & \\ & & M & & \end{array}$$

of R-modules with exact row, it is always possible to find an R-homomorphism $h : B \longrightarrow M$ such that $g = hf$.

Proposition (1.17). Every module M_R can be embedded in an injective right R-module.

Definition (1.18). The minimal injective right R-module E containing M_R is called injective envelope of M and is denoted by E(M).

Remark (1.19). If E is the injective envelope of M then Soc(M) = Soc(E).

Remark (1.20). Every injective module is divisible.

Now we shall define some different types of rings.

Definition (1.21). A ring R is called right (left) hereditary if every right (left) ideal is projective.

Definition (1.22). A ring R is called hereditary if it is both right as well as left hereditary.

Example (1.23) (1) The ring of integers is a hereditary ring. (2) Any principal ideal domain is a hereditary ring.

Definition (1.24). A ring R is called prime ring if (o) is a prime ideal.

Definition (1.25). A ring R is called right Noetherian

(Artinian) if every ascending (descending) chain of right ideals becomes stationary after a finite number of steps.

Definition (1.26). A prime ring which is right hereditary, left hereditary, right Noetherian and left Noetherian is called (hnp)-ring.

Definition (1.27). A ring R is called right (left) bounded if each of its essential right (left) ideal contains a nonzero two sided ideal.

Proposition (1.28) [33, Corollary 4]. Let M be a divisible module over a bounded (hnp)-ring R then M is injective.

Thus over bounded (hnp)-ring divisible module and injective modules are equivalent.

Definition (1.29). In a right R-module M, an element x is said to be a torsion element if xa = o for some regular element a of R, a module whose every element is a torsion element, is called a torsion module.

Proposition (1.30) [33, Lemmas 1,2]. Let R be a bounded (hnp)-ring then the following hold.

(a) Every finitely generated torsion R-module is a direct sum of finitely many uniserial modules.

(b) Any uniform torsion R-module is either of finite length and uniserial or is injective and of infinite length.

(c) Let U and V be two uniform, torsion right R-modules and $b(\neq o) \in U$. If $f : bR \longrightarrow V$ is a non-zero R-homomorphism and length, $d(U/_{bR}) \leq d(V/_{f(bR)})$, then f can be extended to an R-homomorphism $g : U \longrightarrow V$ and $U/_{bR} \cong g(U)/_{g(bR)}$.

(d) Any nonzero homomorphic image of a uniform, torsion R-module is uniform.

Definition (1.31) [33, p.868]. Let M_R be a torsion module over a bounded (hnp)-ring R, then an element $x(\neq o)$ of M is called uniform if xR is a uniform R-module.

Definition (1.32) [33, p.868]. Let M_R be a torsion module over a bounded (hnp)-ring R, then a uniform element $x \in M$ is called of exponent n(denoted by $e(x)$) if $d(xR) = n$; and $\sup \left\{ d(yR/_{xR}) \right\}$, where yR runs over uniform submodules of M containing x, is called the height of x and is denoted by $H_M(x)$ (or simply $H(x)$).

Definition (1.33). Let M_R be a torsion module over a bounded (hnp)-ring R, then M is called bounded if there exists a positive integer k such that $H(x) \leq k$ for all uniform elements $x \in M$.

Proposition (1.34) [33, Lemma 4]. Let M be a torsion module over a bounded (hnp)-ring R and $x_1, x_2, \ldots, x_n$ be finitely many uniform elements of M such that for some non-negative integer k, $H(x_i) \geq k$ for all i. Then for every uniform element x of M in $\Sigma x_i R$, $H(x) \geq k$.

Definition (1.35)[33, p.870]. Let M_R be a torsion module over a bounded (hnp)-ring R then $H_k(M)$ will denote the sub-module of M generated by all those uniform elements of M, which are of height $\geq k$.

Proposition (1.36)[33, Lemma 5]. If $M = U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is a torsion module over a bounded (hnp)-ring R, where each U_i is uniserial, then for any uniform element x of M,

$H(x) \leq \max (d(U_i))-1$ and $e(x) \leq \max (d(U_i))$.

<u>Proposition</u> (1.37)[33,Lemma 6]. If $M = A + B$ is a torsion module over a bounded (hnp)-ring R then for any non-negative integer k, $H_k(M) = H_k(A) + H_k(B)$.

<u>Proposition</u> (1.38) [33,Theorem 3]. Let M_R be a torsion module over a bounded (hnp)-ring R, then M is a direct sum of uniserial submodules (hence cyclic) if and only if M is a union of ascending sequence M_n, ($n = 1,2,\ldots$) of submodules such that for each n, there exists a positive integer k_n, such that $H(x) \leq k_n$ for all uniform elements x of M_n.

<u>Proposition</u> (1.39)[33,Corollary 1]. If M_R is a torsion module over a bounded (hnp)-ring R and P is its socle then M is a direct sum of uniserial modules if and only if P is a union of an ascending sequence $P_n(n = 1,2,\ldots)$ of submodules such that for each n, there exists a positive integer k_n such that $H(x) \leq k_n$ for every uniform element x of P_n.

Let R be an associative ring containing identity and M be an unital right R-module.

Let us consider the following two conditions on M_R as introduced by Singh [34] :

(I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \longrightarrow V$ can be extended to a homomorphism $g : U \longrightarrow V$ provided the composition length $d(U/W) \leq d(V/f(W))$.

M. Zubair Khan has named this module as S_2-module in [28]. As remarked in [34] it can be easily seen [33] that the proofs of Propositions 1.34, 1.36 and 1.37 only depend upon Proposition 1.30(a), (c). <u>Since a module satisfying conditions (I) and (II) satisfies the conditions 1.30(a), (c), hence the Propositions 1.34, 1.36 and 1.37 hold for S_2-modules</u> .

<u>Proposition</u> (1.40)[34, Theorem 1]. Let M be an S_2-module. Then M is a direct sum of uniserial R-submodules if and only if M is a union of ascending sequence $M_n (n = 1, 2, \ldots)$ of submodules such that for each n, there exists a positive integer k_n such that $H_M(x) \leq k_n$ for all uniform elements $x \in M_n$.

The proof of the following can be well adopted from Proposition 1.39.

Corollary (1.41). Let M be an S_2-module. Then M is a direct sum of uniserial modules if and only if $Soc(M)$ is a union of ascending sequence $P_n (n = 1,2,\ldots)$ of submodules such that for each n, there exists a positive integer k_n such that $H(x) \leq k_n$ for each uniform element x of P_n.

Proposition (1.42)[34, Corollary 1]. Any bounded S_2-module is a direct sum of uniserial modules.

Definition (1.43) [2]. An S_2-module M is called S_3-module if it further satisfies the following one more condition:

(III) For any finitely generated submodule N of M, $R/_{ann(N)}$ is right artinian.

Proposition (1.44)[22, Lemma A]. Let M be an S_3-module and N be a uniserial submodule of M having $N = N_o > N_1 > \ldots$ $\ldots > N_t = o$ as its unique composition series. If for $o \leq i \leq t-1$, $P_i = ann\ ({}^{N_i}/_{N_{i+1}})$ then $N_i P_i = N_{i+1}$.

§ 2. h-pure and h-neat submodules

This section is significant in the sense that some of the results, mentioned here, have been very often used in the subsequent chapters. As it is obvious from the very heading of this section, h-pure, h-neat and high submodules are given here and these results have been picked up from [21, 22, 23, 24, 28, 34].

Now we state some useful results whose proofs can be seen in [34].

Proposition (2.1) [34, Theorem 2]. Let M be an S_2-module and N be a submodule of M such that N is a direct sum of uniserial modules of same length k. Then the following are equivalent :

(a) N is a direct summand of M.

(b) $H_n(N) = N \cap H_n(M)$ for all n.

(c) N satisfies $H_k(M) \cap N = o$.

Definition (2.2) [34]. A submodule N of an S_2-module M is called h-pure if $H_n(N) = N \cap H_n(M)$ for all non-negative integer n.

<u>Proposition</u> (2.3) [34, Theorem 3]. A bounded h-pure submodule N of an S_2-module M, is a direct summand of M.

<u>Proposition</u> (2.4) [34, Lemma 1]. Let x be a uniform element in Soc(M) such that $H(x)$ is finite. If $u \in M$ is a uniform element such that $x \in uR$ and $d({}^{uR}/_{xR}) = H(x)$, then uR is a summand of M.

<u>Proposition</u> (2.5) [34, Lemma 2]. Let N be a submodule of an S_2-module M, then the following hold :

(i) If N is h-pure in M, given any uniform element $\bar{x} \in {}^{M}/_{N}$, there exists a uniform element $x' \in M$ such that $e(\bar{x}) = e(x')$ and $\bar{x} = \bar{x}'$.

(ii) If N is h-pure in M, and K is any submodule of N, then $N/_K$ is h-pure in $M/_K$.

(iii) If K is an h-pure submodule of M such that $K \subseteq N$ and $N/_K$ is h-pure in $M/_K$, then N is h-pure in M.

<u>Proposition</u> (2.6) [34, Theorem 4]. Let M be an S_2-module. If every uniform element of Soc(M) is of infinite height, then M is a direct sum of infinite length uniform submodules.

<u>Proposition</u> (2.7) [34, Theorem 5]. Let M be an S_2-module.

Then M has a uniform summand, which can be chosen to be of finite length in case not all uniform elements in $Soc(M)$ are of infinite height.

<u>Proposition</u> (2.8) [21, Proposition 2]. If M is an S_2-module and N is h-pure submodule of M with same socle then $N = M$.

<u>Proposition</u> (2.9) [21, Lemma 2]. If N is an h-pure submodule of an S_2-module M such that $Soc(H_k(M)) \subseteq N$ for some non-negative integer k, then $H_k(M) \subseteq N$.

<u>Proposition</u> (2.10) [28, Lemma 2]. If N is a submodule of an S_2-module M and for every element $x \in Soc(N)$, $H_N(x) = H_M(x)$, then N is h-pure submodule of M.

<u>Proposition</u> (2.11) [28, Theorem 9]. If M is an S_2-module and N is a submodule of M, then N can be embedded in a bounded summand of M if and only if the heights of the uniform elements of N in M are bounded.

<u>Proposition</u> (2.12) [28, Corollary 10]. If M is an S_2-module then every complement of $H_k(M)$ is a direct summand of M.

<u>Definition</u> (2.13) [24]. If M is an S_2-module and N is

a submodule of M then N is called centre of h-purity if every complement of N is h-pure in M.

Proposition (2.14) [24, Corollary 5]. If M is an S_2-module then for every $k \geq o$, $H_k(M)$ is centre of h-purity.

Definition (2.15) [23]. If M is an S_2-module then a submodule N of M is called h-neat if and only if $H_1(N) = N \cap H_1(M)$.

Proposition (2.16) [23, Theorem 3]. A submodule N of an S_2-module M is h-neat if and only if N has no proper essential extension.

Proposition (2.17) [23, Corollary 4]. If M is an S_2-module then h-neat submodule of M coincide with complement submodules.

Definition (2.18) [18]. Let M be an S_2-module. A submodule of $Soc(M)$ is called a subsocle of M.

Definition (2.19) [1]. A subsocle S of an S_2-module M is said to be discrete if $S \cap H_n(M) = o$ for some n.

Definition (2.20) [28]. A submodule N of an S_2-module M is called high submodule if N is maximal disjoint with M^1 where M^1 is the submodule generated by uniform elements of height infinity.

Proposition (2.21) [28, Proposition 20]. Let M be an S_2-module with $M^1 \neq o$ and N, T be high submodules of M then the following hold :

(a) $H_k(T)$ is high submodule of $H_k(M)$ for all k.

(b) $M = T + H_k(M)$ for all k.

(c) $Soc(H_n((T \oplus M^1)/_{M^1})) = Soc(H_n((N \oplus M^1)/_{M^1}))$.

(d) $H_k(M)/_{H_{k+n}(M)} \cong H_k(N)/_{H_{k+n}(N)}$ for all n,k.

(e) $M/_{H_k(N)} = N/_{H_k(N)} \oplus H_k(M)/_{H_k(N)}$ for all k.

(f) M is minimal h-pure module containing $T \oplus M^1$.

Proposition (2.22) [28, Theorem 7]. If M is an S_2-module and N is a submodule of M such that $N \subseteq M^1$. Then any complement T of N is h-pure submodule of M.

§ 3. h-divisible and basic submodules

In this section, we recall some definitions and properties of h-divisible and basic submodules for S_2-modules as introduced by M. Zubair Khan in [26] and [25] respectively.

Definition (3.1) [26]. Let M be an S_2-module then M is called h-divisible if $H_1(M) = M$.

Remark (3.2). An S_2-module M is h-divisible if and only if every uniform element of M is of infinite height.

Proposition (3.3) [26, Lemma 1]. Let M be an S_2-module and $M = \oplus \Sigma M_\alpha$ then M is h-divisible if and only if each M_α is h-divisible.

Proposition (3.4) [26, Lemma 2]. Let M be an S_2-module then M is h-divisible if and only if every uniform element of $Soc(M)$ is of infinite height.

Theorem (3.5) [26, Theorem 3]. If M is an S_2-module then M is h-divisible if and only if M is a direct sum of infinite length uniform submodules.

Proposition (3.6) [26, Theorem 4]. Let M be an S_2-module and N be an h-divisible submodule of M then N is a direct summand of M.

Definition (3.7). An S_2-module M is called reduced if $\{o\}$ is the only h-divisible submodule of M.

<u>Definition</u> (3.8) [25]. Let M be an S_2-module. A submodule B of M is called a basic submodule of M if the folllowing hold:

(1) B is a direct sum of uniserial submodules

(2) B is h-pure in M

(3) $M/_B$ is h-divisible.

The following theorem shows the existence of basic submodules.

<u>Theorem</u> (3.9) [25, Theorem 1]. Let M be an S_2-module then M possesses a basic submodule.

As a characterisation of basic submodules we have the following :

<u>Theorem</u> (3.10) [25, Theorem 2]. Let M be an S_2-module and B be a submodule of M with $B = \oplus \sum_{n=1}^{\infty} B_n$ where each B_n is a direct sum of uniserial modules each of length n. Then B is a basic submodule of M if and only if $M = (B_1 \oplus B_2 \oplus \dots \oplus B_n) \oplus (B_n^* + H_n(M))$ where $B_n^* = B_{n+1} \oplus B_{n+2} \oplus \dots$.

<u>Theorem</u> (3.11) [25, Theorem 3]. Let M be an S_2-module and B be as in Theorem (3.10). Then B is a basic submodule

of M if and only if $B_1 \oplus B_2 \oplus \ldots \oplus B_n$ is a direct summand of M and is maximal with respect to the property $(B_1 \oplus B_2 \oplus \ldots \oplus B_n) \cap H_n(M) = o$.

The following result gives a criterion for submodules to be extended to a basic submodule.

<u>Theorem</u> (3.12) [25, Theorem 4]. A submodule N of an S_2-module M can be extended to a basic submodule B of M if and only if $N = \bigcup_i C_i$ where $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots$, such that the height of uniform elements of C_n (taken in M) are bounded.

The following result gives the uniqueness of basic submodules (upto isomorphism).

<u>Theorem</u> (3.13) [25, Theorem 5]. If M is an S_2-module then any two basic submodules are isomorphic.

CHAPTER - II

h-PURITY AND SOME DECOMPOSITIONS IN S_2-MODULES

The notion of pure subgroup has recently become one of the basic tools in the theory of abelian groups. Therefore, a number of Mathematicians felt the need of generalising this concept for modules. H. Marubayashi did this job for modules over bounded Dedekind prime rings. Analogously Singh [34] generalised the concept of pure subgroups and introduced h-pure submodule for a module satisfying the following two conditions:

(I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f: W \longrightarrow V$ can be extended to a homomorphism $g : U \longrightarrow V$ provided the composition length $d(U/W) \leq d(V/f(W))$.

Recently M. Zubair Khan [28] has named this module as S_2-module and generalised the fundamental results of torsion abelian groups.

The main purpose of this chapter is to further develope the study of h-pure and complement submodules. We have generalised some of the results of abelian groups which, if combined with the results of section 2, Chapter I, will form the foundation stone for the further study of this kind. The basic definitions and other relevant results of S_2-modules are given in section 2, chapter I.

In section 4, some decompositions on S_2-modules have been dealt with. For instance, it is proved that if N is a decomposable submodule of an S_2-module M such that $M/_N$ is bounded, then M is decomposable (Theorem 4.3). We have also obtained a necessary and sufficient condition for an element of an S_2-module to be embedded in a uniform summand of finite length (Theorem 4.10). Also, if M is an S_2-module with elements of infinite height and of finite height, then M possess a non-trivial decomposition (Theorem 4.11). Lastly, we have generalised Prüfer's theorem [15, Theorem 28.1], which gives a useful characterisation of h-purity (Theorem 4.12). In section 5, h-pure-complete modules and hereditary h-pure-complete modules have been studied. We have shown that a decomposable S_2-module is h-pure-complete (Theorem 5.4). We have established an interesting result : if M and K are S_2-modules then $M \oplus K$ is h-pure-complete under some conditions (Theorem 5.7). In section 6, we have studied about the complement submodules. For instance, if N is a submodule of an S_2-module M such that $N \subseteq H_n(M)$ for some n, then for any complement T of N in M, $T \cap H_m(M)$ is an h-neat submodule of $H_m(M)$, for all $m \leq n$ (Theorem 6.2). Towards the end of this section, we have characterised S_2-modules in which every submodule is h-neat (Theorem 6.4).

Some of the results of this chapter are yet to appear in Tamkang J. Math. 11(1980).

§ 4. Some results on decompositions

In this section, we prove some decomposition theorems which will be used in subsequent chapters.

Definition (4.1). If M is an S_2-module then M is called decomposable if M is a direct sum of uniserial submodules.

The following theorem generalises a well known result of abelian groups [15, Theorem 18.1].

Theorem (4.2). If N is a submodule of a decomposable S_2-module M then N is also decomposable.

Proof. Since M is decomposable therefore, by Proposition (1.40), $M = \bigcup_n M_n$, $M_n \subseteq M_{n+1}$ ($n = 1,2,\ldots$) and every uniform element of M_n is of bounded height in M. Now $N=\bigcup_n (N\cap M_n)$ and since for every uniform element $x \in N$, $H_N(x) < H_M(x)$, therefore, every uniform element of $N \cap M_n$ is of bounded height in N. Hence again by Proposition (1.40), N is decomposable.

It is not necessarily true that if a submodule is decomposable then the module itself is decomposable. The following theorem gives a partial characterisation for the decomposability of the module.

Theorem (4.3). If N is a decomposable submodule of an S_2-module M such that $M/_N$ is bounded, then M is decomposable.

Proof. Let $S = Soc(N)$, then by Corollary (1.41) we can write $S = \bigcup_k S_k$ where $S_k \subseteq S_{k+1}$ such that for every positive integer k, there exists a positive integer t_k such that $H_N(x) \leq t_k$ for every uniform element $x \in S_k$. Let $Soc(M) = S \oplus X$. As $M/_N$ is bounded, there exists a positive inter t

such that $H_{M/N}(\bar{x}) \leq t$ for every uniform element $\bar{x} \in M/_N$. Now $Soc(M) = \bigcup_k (S_k + X)$ and for any uniform element $y \in S_k+X$, $H_M(y) \leq t_k + t$. Hence again by Corollary (1.41), M is decomposable

Corollary (4.4). Let M be an S_2-module and k be a positive integer, then M is decomposable if and only if $H_k(M)$ is decomposable.

Proof. If M is decomposable then by Theorem (4.2), $H_k(M)$ is decomposable. Suppose $H_k(M)$ is decomposable then as $M/_{H_k(M)}$ is bounded, therefore by Theorem (4.3), M is decomposable.

It is trivial that for any uniform element $x \in N \subseteq M$, $H_N(x) \leq H_M(x)$. In the following result we use the decomposability condition to establish the condition underwhich the height of a uniform element in a submodule is strictly less than its height in the module.

Theorem (4.5). Let M be an S_2-module which is not decomposable. If K is minimal submodule of M such that both K and $M/_K$ are decomposable, then for every uniform element $x \in Soc(K)$, $H_K(x) < H_M(x)$.

<u>Proof</u>. Suppose x is a uniform element in $Soc(K)$ such that $H_K(x) = H_M(x)$. Since K is a direct sum of uniserial submodules, then by Proposition (1.40), it follows that $H_K(x)$ is finite say n. Then there exists a uniform element $y \in K$ such that $x \in yR$ and $d({}^{yR}/_{xR}) = n$. Therefore, by Proposition (2.4), yR is a direct summand of M i.e. $M = yR \oplus N$. Then obviously $K = yR \oplus N \cap K$. Now we show that ${}^{M}/_{N \cap K}$ is a direct sum of uniserial submodules. For this, we have

$$
\begin{aligned}
{}^{M}/_{N \cap K} &= (yR \oplus N)/_{N \cap K} \\
&= (yR + N \cap K)/_{N \cap K} \oplus {}^{N}/_{N \cap K} \\
&= yR \oplus (N+K)/_{K} ,
\end{aligned}
$$

where $(N+K)/_K$, being submodule of ${}^{M}/_{K}$, is also direct sum of uniserial submodules. Also $N \cap K$, being submodule of K, is direct sum of uniserial submodules. Hence ${}^{M}/_{N \cap K}$, $N \cap K$ are decomposable modules and this contradicts the minimality of K. Hence the result follows.

<u>Corollary</u> (4.6). Let M be an S_2-module which is not

decomposable. If K is minimal submodule of M such that both K and $M/_K$ are decomposable, then K contains no non-zero h-pure submodule of M.

Proof. Let x be any uniform element in $Soc(K)$, then by Theorem (4.5), $H_K(x) < H_M(x)$ and hence K is not h-pure submodule of M. Thus, K does not contain non-zero h-pure submodule of M.

Theorem (4.7). Let M be an S_2-module and $M^1 = o$. If K is a non-zero submodule of M such that $M/_K$ is decomposable then there exists a submodule N of K such that $N \neq K$ and $M/_N$ is decomposable.

Proof. Since $K \neq o$ and $M^1 = o$, then there exists a uniform element $x \in Soc(K)$ such that $H_M(x) = n < \infty$. Therefore, there exists a uniform element $y \in M$ such that $x \in yR$ and $d(^{yR}/_{xR}) = n$. Thus, by Proposition (2.4), yR is a direct summand of M i.e. $M = yR \oplus T$. Let $N = K \cap T$ then $N \subset K$ and $N \neq K$, as $x \notin N$. Also, $M/_N = (yR \oplus T)/_N \cong (yR + N)/_N \oplus T/_N \cong$ $yR/_{yR \cap N} \oplus T/_N = yR \oplus T/_N$. Now, $T/_N = T/_{K \cap T} \cong (K+T)/_T$ and $(T+K)/_T$, being a submodule of $M/_T$, is decomposable. Hence

$M/_N$ is decomposable.

The following theorem which is a generalisation of a result [11] of K.Benabdallah, J. M. Irwin and M. Rafiq, characterises completely the minimality of submodule K of M with respect to the property that both K and $M/_K$ are decomposable. Since the proof is very much on similar lines, it is omitted.

Theorem (4.8). Let M be an S_2-module which is not decomposable and K be a submodule of M such that K and $M/_K$ are decomposable. Then K is minimal with respect to these properties if and only if $K = M^1$.

The following theorem is a useful result for future use in the subsequent articles.

Theorem (4.9). An S_2-module M is a direct sum of uniserial submodules if and only if M is union of an ascending chain $M_n(n = 1,2,\ldots)$ of bounded h-pure submodules.

Proof. Suppose M is decomposable then by Proposition (1.40), M is a union of ascending sequence $M_n(n = 1,2,\ldots)$ of submodules such that for every uniform element $x \in M_n, H_M(x) \leq k_n$

where $k_n \geq o$. Now using Proposition (2.11), each M_n can be embedded in a bounded summand K_n of M and $M = \cup K_n$, where K_n are h-pure in M. Conversely, let M be a union of an ascending chain of bounded h-pure submodules then by Proposition (2.3), M is decomposable.

The following theorem generalises [14, Theorem 24.7].

<u>Theorem</u> (4.10). Let M be an S_2-module. An element $b \in M$ can be embedded in a uniform summand of finite length if and only if $bR \cap M^1 = o$.

<u>Proof</u>. Suppose that b can be embedded in a uniform summand N of length k in M, then every uniform element in bR will be of height $< k$ in N and hence in M as N is h-pure in M. Therefore, $bR \cap M^1 = o$. Conversely, suppose that $bR \cap M^1 = o$. Now $bR = \oplus \sum_{i=1}^{m} x_i R$, where each $x_i R$ is uniserial submodule of finite length k_i. Let $H_{bR}(x_i) = n_i$, $k = \max\{k_i\}$ and $n = \max\{n_i\}$. Therefore, height of each uniform element of bR in M is atmost $k+n$. Then by Proposition (2.11), bR can be embedded in a bounded direct summand of M.

As shown by Singh [34, Corollary 1], a bounded S_2-module is a direct sum of uniserial modules. Also, M. Zubair Khan [26, Theorem 3] proved that an S_2-module M is a direct sum of infinite length uniform submodules if and only if every uniform element of M is of infinite height. Now it is very natural to consider the question : If M is an S_2-module with elements of infinite height and of finite height, then does M possess a non-trivial decomposition ? The following result gives an answer to it in a particular case.

<u>Theorem</u> (4.11). Let M be an S_2-module with elements of infinite height and of height $< k$. Then $M = (\oplus \Sigma\, x_i R) \oplus (\oplus \Sigma\, U_j)$, where $x_i R$ are uniserial submodules of finite length and U_j are uniform submodules of infinite length.

<u>Proof</u>. Trivially, every uniform element of $H_k(M)$ is of height $\geq k$. Hence, by assumption, every uniform element of $H_k(M)$ is of infinite height. Therefore, appealing to Proposition (3.6), we get $M = H_k(M) \oplus T$. Now again appealing to Proposition (1.42) and Theorem (3.5), the assertion follows.

The following theorem, a generalisation of Prüfer's theorem [15, Theorem 28.1], is a very useful characterisation of h-purity.

Theorem (4.12). A submodule N of an S_2-module M is h-pure if and only if given any uniform element $\bar{x} \in M/_N$ there exists a uniform element $x' \in M$ such that $e(\bar{x}) = e(x')$ and $\bar{x} = \bar{x}'$.

Proof. The necessity follows from Proposition (2.5). For the sufficiency, let n be the smallest positive integer such that $H_n(N) \neq N \cap H_n(M)$. We can find a uniform element x of smallest exponent such that $x \in N \cap H_n(M)$ but $x \notin H_n(N)$ then $H_N(x) = n-1$. Now we can find a uniform element $y \in M$ such that $x \in yR$ and $d(yR/_{xR}) = n$. Let $zR/_{xR} = Soc({}^{yR}/_{xR})$ then $d({}^{yR}/_{zR}) = n-1$ and thus $z \in H_{n-1}(M)$. If $z \in N$, we shall get $z \in H_{n-1}(N)$ and as a result $x \in N$, a contradiction. Hence $z \notin N$. Consequently $x = yR \cap N$. So by hypothesis there exists a uniform element $y' \in M$ such that $\bar{y}' = \bar{y}$ and $e(y') = e(\bar{y})$, then $y'R \cap N = o$ and the map $f : \bar{y}R \longrightarrow y'R$ given by $f(\bar{y}r) = y'r$ is an isomorphism. The composition of f with the natural homomorphism $yR \longrightarrow \bar{y}R$ induces an R-epimorphism $g: yR \longrightarrow y'R$, given by $g(yr) = y'r$. Now $\ker g = xR$, $x = yt$ for some $t \in R$, $y-y'$ is uniform and $x = (y-y')t$. Clearly $y-y' \in N$ is uniform and $d({}^{(y-y')R}/_{xR}) = n$ which gives $x \in H_n(N)$. This is a contradiction. Hence the result follows.

§ 5. h-pure-complete modules

This section is devoted mainly for the study of h-pure-complete modules. We have also introduced the concept of hereditary h-pure-complete modules.

Definition (5.1). A subsocle S of an S_2-module M is said to support a submodule N of M if and only if $Soc(N) = S$.

As in [25], an S_2-module M is said to be h-pure-complete if for every subsocle S of M there exists an h-pure submodule N of M such that $S = Soc(N)$.

Lemma (5.2). Let M be an S_2-module and S be a submodule of $Soc(M)$ such that $S \cap H_n(M) = o$, for some n. Then there exists an h-pure submodule K of M such that $Soc(K) = S$.

Proof. Let N be the complement of $H_n(M)$ in M then by Proposition (2.12), N is a direct summand of M and hence N is h-pure in M. Also N is bounded containing S. We have $S = \Sigma x_i R$. Since N is bounded then $H_N(x_i) = k_i < \infty$. Therefore, there exists a uniform element $y_i \in N$ such that $x_i \in y_i R$ and

$d({}^{y_iR}/_{x_iR}) = k_i$. Now by Proposition (2.4), y_iR is an h-pure submodule of N. Thus, each x_iR can be embedded in an h-pure submodule y_iR of N such that $Soc(y_iR) = x_iR$. Hence $S = \Sigma\, x_iR$ can be embedded into an h-pure submodule $\Sigma\, y_iR = K$ of N such that $Soc(K) = S$.

The following proposition, a generalization of a result of P. Hill and C. Megibben [31, Proposition 3], establishes the condition under which a subsocle supports an h-pure submodule.

<u>Proposition</u> (5.3). Let M be an S_2-module and let $S = \cup S_n$ be the union of ascending chain of subsocles S_n of M. If $S_n \cap H_n(M) = o$, for $n = 1,2,\ldots$, then S supports an h-pure submodule of M.

<u>Proof.</u> For any S_n, we have by Lemma (5.2), S_n supports an h-pure submodule K_n of M such that $K_n \cap H_n(M) = o$. Now we show that $(K_n + S_n) \cap H_{n+1}(M) = o$. For this we have $Soc(K_n + S_{n+1}) = Soc(K_n) + S_{n+1} \subseteq S_{n+1}$. But $S_{n+1} \cap H_{n+1}(M) = o$ then $Soc(K_n + S_{n+1}) \cap H_{n+1}(M) = o$. This implies that $(K_n + S_{n+1}) \cap H_{n+1}(M) = o$. Hence $K_n + S_{n+1}$ can be embedded into a complement T_{n+1} of $H_{n+1}(M)$ in M, then by Proposition(2.12)

T_{n+1} is a bounded direct summand of M, containing K_n+S_{n+1}. As K_n is bounded h-pure submodule of T_{n+1} then by Proposition (2.3), K_n is a direct summand of T_{n+1} i.e. $T_{n+1}= K_n \oplus N_{n+1}$. Then we assert that $S_{n+1} = Soc(K_n) \oplus (N_{n+1} \cap S_{n+1})$. For this let x be a uniform element in S_{n+1}, we have $x \in Soc(T_{n+1}) = Soc(K_n) \oplus Soc(N_{n+1})$ so that $x = y+z$, where $y \in Soc(K_n)$ and $z \in Soc(N_{n+1})$. Then $x-y = z \in S_{n+1}$. This gives that $z \in S_{n+1} \cap N_{n+1}$ and hence $S_{n+1} \subseteq Soc(K_n) \oplus (N_{n+1} \cap S_{n+1})$. Now, N_{n+1} is bounded, therefore, $(N_{n+1} \cap S_{n+1}) \cap H_k(N_{n+1}) = o$ for some k. Hence, again by Lemma (5.2), $N_{n+1} \cap S_{n+1}$ supports an h-pure submodule L_{n+1} in N_{n+1}. Let us write $K_{n+1}= K_n \oplus L_{n+1}$, then the h-pure submodule K_{n+1} is supported by S_{n+1} in M. Hence the union $K = \cup K_n$ of the ascending chain of h-pure submodules $K_1 \leq K_2 \leq \ldots \leq K_n \leq \ldots$ of M is an h-pure submodule of M supported by the union $\cup S_n = S$.

As a consequence of the above result, now we show that every decomposable S_2-module is h-pure-complete.

<u>Theorem</u> (5.4). A decomposable S_2-module M is h-pure-complete.

<u>Proof.</u> Let $M = \oplus \Sigma M_n$ be a direct sum of uniserial submodules. We denote K_n the direct sum of the uniserial summands of M each of length n then we can write $M = \oplus \Sigma K_n$. Then by Proposition (2.1), $K_n \cap H_n(M) = o$. If S is a subsocle of M, then taking $S_n = (K_1 \oplus K_2 \oplus \dots \oplus K_n) \cap S$, we have $S_1 \leq S_2 \leq \dots \leq S_n \leq \dots$ as an ascending chain of subsocles of M such that $S = US_n$. Also, $S_n \cap H_n(M) = o$. Then by Proposition (5.3), S supports an h-pure submodule of M. Hence M is h-pure-complete.

<u>Lemma</u> (5.5). Let K be the submodule of an S_2-module M, then $H_n(M/K) = (H_n(M) + K)/_K$ for all n.

<u>Proof.</u> Trivially, $(H_n(M) + K)/_K \subseteq H_n(M/_K)$. Now suppose $\bar{x} \in H_n(M/_K)$ be a uniform element. Then there exists a uniform element $\bar{y} \in M/_K$ such that $\bar{x} \in \bar{y}R$ and $d(\bar{y}R/_{\bar{x}R}) = n$. We can assume that y itself is uniform. Now $\bar{x} = \bar{y}r$ for some $r \in R$. If $yr = y'$, then $yrR = y'R \subseteq yR$ implies that $d(^{yR}/_{yrR}) = n$ and hence $\bar{y}r \in (H_n(M) + K)/_K$ i.e. $H_n(M/_K) \subseteq (H_n(M) + K)/_K$. Therefore, $H_n(M/_K) = (H_n(M)+K)/_K$.

<u>Proposition</u> (5.6). Let M,K be S_2-modules with K bounded

and S be a subsocle of $M \oplus K$. Let N be an h-pure submodule of M supported by $S \cap M$. Then S supports an h-pure submodule of $M \oplus K$ which contains N.

Proof. Using Lemma (5.5), we have

$$\begin{aligned}((S+N)/_N) \cap H_n((M \oplus K)/_N) &= ((S+N)/_N) \cap (H_n(M \oplus K)+N)/_N \\ &= ((S+N)/_N) \cap (H_n(M)+N)/_N \\ &= 0, \text{ since } S \cap M \text{ supports } N.\end{aligned}$$

Therefore, $(S+N)/_N$ is a discrete subsocle of $(M \oplus K)/_N$ which yields that $(S+N)/_N$ supports an h-pure submodule $N'/_N$ of $(M \oplus K)/_N$ and therefore, $(S+N)/_N = Soc(N'/_N)$. Now we assert that $S = Soc(N')$. Trivially, if $x \in S$ then $x \in Soc(N')$. Let y be a uniform element in $Soc(N')$ with $y \notin N$, then $\bar{y} \in Soc(N'/_N) = (S+N)/_N$ and hence $y \in S$. Therefore, $S = Soc(N')$. Since N is h-pure submodule of M and M is h-pure in $M \oplus K$. Thus N is h-pure in $M \oplus K$. Also, $N'/_N$ is h-pure in $(M \oplus K)/_N$, therefore, N' is h-pure submodule of $M \oplus K$.

As shown by P.Hill and C.Megibben [32], the direct sum of two h-pure-complete modules need not be h-pure-complete.

However, the following theorem which is a generalisation of a result of P. Hill and C. Megibben [32, Theorem 5.2] shows that the direct sum is h-pure-complete under some conditions.

Theorem (5.7). Suppose M and K are S_2-modules. If M is h-pure-complete and K is decomposable, then $M \oplus K$ is h-pure-complete.

Proof. Using Theorem (4.9), K can be written as a union of an ascending chain of bounded h-pure submodules $K_1 \leq K_2 \leq \ldots$ $\ldots \leq K_n \leq \ldots$. Let S be a subsocle of $M \oplus K$ and $S_n = (M \oplus K_n) \cap S$. Since M is h-pure-complete then all subsocles $S_n \cap M$ of M will support h-pure submodules N_n. Therefore, by Proposition (5.6), we can find an ascending chain of h-pure submodules T_n of $M \oplus K_n$, containing N_n, supported by S_n. Thus the union $\cup T_n = T$ will be an h-pure submodule of $M \oplus K$ supported by $\cup S_n = S$. Hence $M \oplus K$ is h-pure-complete .

Definition (5.8). An S_2-module M is called hereditary h-pure-complete if every submodule of M is h-pure-complete.

Theorem (5.9). If M is a decomposable S_2-module then M is hereditary h-pure-complete.

Proof. It is clear from Theorem (4.2) that every submodule of a decomposable S_2-module is decomposable. Now using Theorem (5.4), M is hereditary h-pure-complete.

§ 6. Complement submodules

Proposition (2.17) shows that the h-neat and complement submodules coincide in S_2-modules. The main purpose of this section is to obtain the S_2-module in which every submodule is h-neat.

The following theorem generalises a result of K.Simauti [12] which shows that a complement submodule is how much near to h-pure submodules.

Theorem (6.1). Let M be an S_2-module. If N is a submodule of M such that $N \subseteq H_n(M)$ for some n. Then for any complement T of N in M,

$$T \cap H_k(M) = H_k(T) \quad \text{for every} \quad k \leq n .$$

Proof. Obviously, $T \cap H_1(M) = H_1(T)$. Suppose that for some $m < n$, $T \cap H_m(M) = H_m(T)$. Let $x \in T \cap H_{m+1}(M)$ be a uniform element. Then there exists a uniform element $y \in M$

such that $x \in yR$ and $d({}^{yR}/_{xR}) = 1$ and so $y \in H_m(M)$. If $y \in T$ then $y \in H_m(T)$. Consequently, $x \in H_{m+1}(T)$. If $y \notin T$ then $(T + yR) \cap N \neq o$. Hence, there exists a uniform element $z \in N$ such that $z = t + yr$ where $t \in T$ and $r \in R$. As yR is totally ordered, either $xR < yrR$ or $yrR \leq xR$. But $xR < yrR$ is not possible as $d({}^{yR}/_{xR}) = 1$. Also, $yrR \leq xR$ implies that $yrR \subseteq T$ yields $yr \in T$. Therefore, $z \in T$, a contradiction. Hence $yrR = yR$. Now, without any loss of generality, we can assume that $z = t + y$. Define $f : yR \longrightarrow tR$ given as $yr \longrightarrow tr$ then as T is the complement of N, it is easy to check that f is a well defined onto homomorphism. Hence t is uniform. Since $x = yr_o$ for some $r_o \in R$ and $zr_o = tr_o + yr_o$. Also, it is easy to see that $zr_o = o$. Hence $x = yr_o = - tr_o$.

Now we assert that $tr_oR < tR$. For, if $tr_oR = tR$, then $t = tr_or'$ for some $r' \in R$ and we get $z = yr_1$ for some $r_1 \in R$. Obviously, $zR < yR$ and again either $xR \subseteq zR$ or $zR \subseteq xR$. But none is possible. Hence $tr_oR < tR$. Also $t = z-y \in N + H_m(M) \subseteq H_m(M) + H_m(M) \subseteq H_m(M)$. So that $t \in T \cap H_m(M) = H_m(T)$. Therefore, $x = - tr_o \in H_{m+1}(T)$.

Hence the result follows.

Theorem (6.2). Let M be an S_2-module and N be a submodule of M such that $N \subseteq H_n(M)$ for some n. Then for any complement T of N in M, $T \cap H_m(M)$ is an h-neat submodule of $H_m(M)$, for all $m \leq n$.

Proof. Let $x \in H_m(M)$ be a uniform element such that $x \notin T \cap H_m(M)$. Suppose that $(T \cap H_m(M) + xR) \cap N = o$. Since $x \notin T$, we get $(T + xR) \cap N \neq o$ so that there exists a uniform element $y \in N$ such that $y = t + xr$, where $t \in T$ and $r \in R$. Then $t = y - xr \in N + H_m(M) \subseteq H_n(M) + H_m(M) \subseteq H_m(M)$. Therefore, $t \in T \cap H_m(M)$. Consequently, $y \in T \cap H_m(M) + xR$. This gives that $y \in (T \cap H_m(M) + xR) \cap N$, a contradiction. Hence $T \cap H_m(M)$ is the complement of N in $H_m(M)$. Thus, by Proposition (2.17), $T \cap H_m(M)$ is h-neat in $H_m(M)$.

Lemma (6.3). Let M be an S_2-module. Then for any uniform element $x \in M$, xR is h-neat in M if and only if $\underset{M}{H}(x) = o$.

Proof. If xR is h-neat in M then $xR \cap H_1(M) = H_1(xR)$. Let $H_M(x) \geq 1$, then $x \in xR \cap H_1(M) = H_1(xR)$ which yields that $d(^{xR}/_{xR}) = 1$, which is not possible. Hence $H_M(x) = o$. Conversely, suppose that $H_M(x) = o$. Let $y \in xR \cap H_1(M)$ be a uniform element. Then there exists a uniform element $z \in M$ such that $y \in zR$ and $d(^{zR}/_{yR}) = 1$. If $yR = xR$, then $H_M(x) \neq o$, a contradiction. Hence $yR < xR$. Therefore, $y \in H_1(xR)$ and so $xR \cap H_1(M) = H_1(xR)$.

K. Kato [9] has given a characterisation of those abelian groups in which every subgroup is neat. Analogous to this we have got the following simple characterisation of S_2-modules in which every submodule is h-neat.

<u>Theorem</u> (6.4). Let M be an S_2-module. Then $M = Soc(M)$ if and only if every submodule of M is h-neat in M.

<u>Proof.</u> Suppose that $M = Soc(M)$, then for any submodule N of M, $N \cap H_1(M) = o = H_1(N)$. Conversely, suppose that every submodule of M is h-neat in M. Then for any uniform element $x \in M$, xR is h-neat in M. Therefore, by Lemma (6.3), $H_M(x)=o$ and hence $M = Soc(M)$.

CHAPTER - III

h-DENSE AND LARGE SUBMODULES

The notion of large subgroups of an abelian group was first introduced by R. S. Pierce. He investigated the relation between the structure of primary abelian groups and their large subgroups. The large subgroup is one of the important structure in abelian groups. Since then a number of group theorist like C. Megibben, K. M. Benabdallah, B.J. Eisenstadt, J. M. Irwin, E. V. Poluianov and M. Zubair Khan did a tremendous work on the various properties and representation of large subgroups. Due to the importance of this notion a need was felt to generalise this notion for S_2-modules and to some extent we have shown that a number of results, true for torsion abelian groups, are also valid for S_2-modules. In section 7, we have introduced the concept of h-dense submodule for S_2-module and obtained some useful results (Proposition 7.2, Theorem 7.5, Proposition 7.6). Introducing h-dense subsocle we have shown that an h-neat submodule of an S_2-module supported by an h-dense subsocle is h-pure (Theorem 7.9). Section 8 deals with the study of large submodules. It is proved that if L is a large submodule of an S_2-module M then $L^1 = M^1$ (Theorem 8.4). Theorem (8.7) discusses the decomposability of the module in terms of its large submodules. Lastly, in section 9, closure of h-dense and large submodules have been studied. In [19], M. Zubair Khan has obtained the structure of those subgroups whose closures are always large. Analogous to this we have characterised the submodules of S_2-module whose closures are large (Theorem 9.5). Lastly, it is shown that a high submodule of a large submodule is closed in a high submodule of the module.

§ 7. h-dense submodules

In this section, we have defined h-dense submodule and obtained some necessary and sufficient conditions for a module to be h-dense. Further, we have also defined h-dense subsocles and established an interesting result that an h-neat submodule of an S_3-module supported by an h-dense subsocle is h-pure.

Definition (7.1). A submodule N of an S_2-module M is called h-dense if and only if $M/_N$ is h-divisible.

Proposition (7.2). A submodule N of an S_2-module M is h-dense if and only if $M = N + H_n(M)$ for every n.

Proof. If N is h-dense in M then $M/_N$ is h-divisible. Therefore, $M/_N = H_n(M/_N) = (H_n(M) + N)/_N$ by Lemma (5.5). So that $M = N + H_n(M)$. Conversely, let $M = N + H_n(M)$ then $M/_N = (N + H_n(M))/_N = H_n(M/_N)$. Thus, N is h-divisible and hence the result follows.

As a consequence of the above proposition, we have

Corollary (7.3). If N is h-dense in an S_2-module M, then every submodule K with $N \subseteq K \subseteq M$ is also h-dense in M.

Now, we are in a position to restate the definition (5.8) in another form as ; a submodule B of an S_2-module M is called a basic submodule of M if and only if the following hold ;

(i) B is decomposable

(ii) B is h-pure in M

(iii) B is h-dense in M.

The following proposition is a generalisation of a result of Fuchs [14] which gives some interesting properties of basic submodules.

Proposition (7.4). Let M be an S_2-module and B be a basic submodule of M then the following hold :

(a) $Soc(M/_B) \cong Soc(M)/_{Soc(B)}$,

(b) $Soc(M/_B) \cong Soc(H_n(M)/_{H_n(B)})$ for every n,

(c) $Soc(H_n(M)/_{H_n(B)}) \cong Soc(H_n(M))/_{Soc(H_n(B))}$ for every n,

(d) $Soc(H_n(M)) = Soc(H_n(M)) + Soc(H_{n+1}(M))$ for every n,

(e) $Soc(H_{n+1}(B)) = Soc(H_n(B)) \cap Soc(H_{n+1}(M))$ for every n,

(f) $Soc(H_n(B))/_{Soc(H_{n+1}(B))} \cong Soc(H_n(M))/_{Soc(H_{n+1}(M))}$ for every n.

Proof (a). Firstly we show that $Soc(M/_B) = (Soc(M)+B)/_B$. Trivially, we have $(Soc(M)+B)/_B \subseteq Soc(M/_B)$. Let $\bar{x} \in Soc(M/_B)$ be a uniform element then $e(\bar{x}) = 1$ and $\bar{x} \in M/_B$. Therefore, by Proposition (2.5), there exists a uniform element $y \in Soc(M)$ such that $\bar{x} = \bar{y}$ so that $\bar{y} \in (Soc(M)+B)/_B$ and hence $Soc(M/_B) \subseteq (Soc(M)+B)/_B$. Thus, $Soc(M/_B) = (Soc(M)+B)/_B$. Now $(Soc(M)+B)/_B \cong Soc(M)/_{B \cap Soc(M)} = Soc(M)/_{Soc(B)}$. Hence the result follows.

(b). Since $M/_B$ is h-divisible, $M/_B = H_n(M/_B) = (H_n(M)+B)/_B$ by Lemma (5.5), so that $M/_B \cong H_n(M)/_{B \cap H_n(M)} = H_n(M)/_{H_n(B)}$ and hence $Soc(M/_B) \cong Soc(H_n(M)/_{H_n(B)})$ for every n.

The proofs of (c), (d), (e) and (f) can be well adopted.

Now we prove the following.

<u>Theorem</u> (7.5). If N is an h-pure submodule of an S_2-module M, then N is h-dense in M if and only if N contains a basic submodule of M.

<u>Proof</u>. Suppose N is h-dense in M. Let B be a basic submodule of N then B is h-pure in M. Also $N/_B$ is h-divisible submodule of $M/_B$ then by Proposition (5.6), $N/_B$ is a direct summand of $M/_B$ i.e. $M/_B = N/_B \oplus K/_B$ and $K/_B \cong M/_N$ is h-divisible. Hence $M/_B$ is also h-divisible. Thus B is a basic submodule of M.

Conversely, suppose that N contains a basic submodule B of M. Then $M/_N \cong M/B/_{N/_B}$ is h-divisible and hence N is h-dense in M.

The proof of the following can be well adopted from the above theorem.

<u>Proposition</u> (7.6). Let M be an S_2-module. Then a submodule N containing a basic submodule B of M is h-pure if and only if B is h-dense in N.

The following proposition, a generalisation of a result of Megibben [4] can be proved easily.

<u>Proposition</u> (7.7). If K is minimal h-pure submodule of an S_2-module M containing a submodule N of M such that there is a submodule L of N which is h-dense in N and h-pure in M, then $K = N$.

<u>Definition</u> (7.8). Let M be an S_2-module and S be a sub-socle of M. Then S is called h-dense subsocle if $Soc(M) = S + Soc(H_k(M))$ for every $k \geq 0$.

Now we have the following main theorem which is a generalisation of a result of P.Hill and C. Megibben [31].

<u>Theorem</u> (7.9). An h-neat submodule of an S_3-module supported by an h-dense subsocle is h-pure.

<u>Proof.</u> Let N be an h-neat submodule of M then $N \cap H_1(M) = H_1(N)$. Now suppose $H_n(N) = N \cap H_n(M)$ for some n. Let x be a

uniform element in $N \cap H_{n+1}(M)$ then we can find a uniform element $y \in N$ such that $x \in yR$ and $d({}^{yR}/_{xR}) = n+1$. Let ${}^{zR}/_{xR} = \mathrm{Soc}({}^{yR}/_{xR})$. If $z \in N$ then there is nothing to prove. Let $z \notin N$. Since $d({}^{zR}/_{xR}) = 1$, we can find a uniform element $u \in N$ such that $x \in uR$ and $d({}^{uR}/_{xR}) = 1$. Now appealing to condition (II), there exists an isomorphism $f : zR \longrightarrow uR$ such that f is the identity on xR. Trivially, the map $g : zR \longrightarrow (z - f(z))R$, given as $zr \longrightarrow (z-f(z))r$, is an R-epimorphism with $xR \subseteq \ker g$. Hence $e(z-f(z)) \leq 1$ and we get $z-f(z) = z-u \in \mathrm{Soc}(M)$. Since S is h-dense subsocle of M, $z - u - s \in H_n(M)$ for some $s \in S$ and $z-u-s = t$ for some $t \in H_n(M)$, Now by supposition $z-t = u+s \in H_n(N)$. Now $(u+s)R = \oplus \Sigma\, b_iR$ where $b_i \in H_n(N)$. Trivially b_i can not be of exponent 1. Similarly $sR = \oplus \Sigma\, t_iR$ where t_iR are simple modules. Let $P_i = \mathrm{ann}\,(t_iR)$ and $P = \mathrm{ann}\,({}^{uR}/_{xR})$ then by Proposition (1.44), $sP_1P_2 \ldots P_q = o$ and $uP = xR$. Let $b_1, b_2, \ldots, b_\alpha$ be uniform elements of exponent greater than 1 and $b_{\alpha+1}, \ldots, b_n$ be uniform elements of exponent 1. Now we can find submodules w_jR such that $d({}^{b_jR}/_{w_jR}) = 1$. Let $Q_j = \mathrm{ann}\,({}^{b_jR}/_{w_jR})$ then $b_jQ_j = w_jR$ for $j = 1,2,\ldots,\alpha$. Let

$Q_i = \text{ann}(b_iR)$, $i = \alpha+1,\dots,n$ then $b_iQ_i = o$. Without any loss of generality we can assume $P_1, P_2,\dots,P_q, Q_1,\dots,Q_\alpha, P$ to be distinct. Now

$$(u+s)RP_1P_2 \dots P_qQ_1 \dots Q_\alpha Q_{\alpha+1} \dots Q_nP$$

$$= uP_1 \dots P_qQ_1\dots Q_\alpha Q_{\alpha+1} \dots Q_nP = uP = xR.$$

Also,

$$(u+s)RP_1P_2\dots P_qQ_1\dots Q_\alpha Q_{\alpha+1}\dots Q_nP = \sum_{i=1}^{\alpha} b_iP_1\dots P_qQ_1\dots Q_nP,$$

but xR is uniform, hence $xR = b_jP_1\dots P_qQ_1\dots Q_\alpha Q_{\alpha+1}\dots Q_nP \subseteq w_jR < b_jR$ and we get $d({}^{b_jR}/_{xR}) \geq 1$. Therefore, $x \in H_{n+1}(N)$. Hence by induction, N is h-pure submodule of M.

§ 8. Large submodules

The concept of large subgroups was first introduced by R. S. Pierce. Latter in [10], K. M. Benabdallah, B.J.Eisenstadt, J. M. Irwin and E. W. Poluianov investigated the relation between the structure of primary abelian groups and their large subgroups. Analogous to this notion, we have defined large submodules and studied various properties of large submodules. We denote by M^1 as the submodule of M generated by the uniform elements of height infinity.

Definition (8.1). A submodule L of an S_2-module M is said to be large if L is fully invariant and $M = L + B$, for every basic submodule B of M.

Proposition (8.2). Let M be an S_2-module, then $H_n(M)$ is a large submodule of M for every n.

Proof. The proof is immediate from Proposition (7.2).

Proposition (8.3). If L is a large submodule of an S_2-module M, then $H_n(L)$ is also large submodule of M for every n.

Proof. Using Proposition (8.2), we have

$$\begin{aligned} M &= H_n(M) + B \\ &= H_n(L+B) + B \\ &= H_n(L) + B. \end{aligned}$$

Therefore, $H_n(L)$ is large in M.

Now we prove the following useful theorem.

Theorem (8.4). If L is a large submodule of an S_2-module M, then $L^1 = M^1$.

Proof. Let $x \in M^1$ be a uniform element then $x = b+y$ where $y \in L$ and $b \in B$, for some basic submodule B of M. Since, $bR \cap M^1 = o$, therefore, by Theorem (4.10), b can be embedded into a finite length summand B_1 of M. Thus $M = B_1 \oplus N$. Let $\pi : M \longrightarrow N$ be a projection then $\pi(b) = o$, so $\pi(x) = \pi(y) \in L$. Now, $x = \pi(x) + (I-\pi)(x)$ yields $(I-\pi)(x)$ is a uniform element of B_1 of infinite height. Hence $(I-\pi)(x) = o$. Consequently, $x = \pi(x) = \pi(y) \in L$. Thus, M^1 is contained in every large submodule of M. Since by Proposition (8.3), $H_n(L)$ is large submodule of M, we have $L^1 \subseteq M^1 \subseteq \bigcap_{n=1}^{\infty} H_n(L) = L^1$. Hence, $L^1 = M^1$.

The proof of the following interesting proposition can be well adopted from [16, Proposition 67.4].

Proposition (8.5). If L is a large submodule of an S_2-module M, then $M/_L$ is decomposable.

Now we consider an S_2-module M satisfying the following condition ;

(A) For every large submodule L of M, $Soc(L)=Soc(H_n(M))$ for some n.

Now in order to prove the main theorem of this section, we need the following :

Proposition (8.6). Let M be an S_2-module satisfying the condition (A), N be a submodule of M and L be a large submodule of M. Then there exists an n such that $N \cap H_n(L) = o$ if and only if there exists an m such that $N \cap H_m(M) = o$.

Proof. Suppose $N \cap H_m(M) = o$ for some m, then the assertion trivially holds. Now $N \cap H_n(L) = o$ for some n. Since L is a large submodule of M then by Proposition (8.3), $H_n(L)$ is also large submodule of M. As, $Soc(H_n(L)) = Soc(H_m(M))$ for some m, we have

$$Soc(N \cap H_m(M)) = Soc(N) \cap Soc(H_m(M))$$

$$= Soc(N) \cap Soc(H_n(L)) = o.$$

Therefore, $N \cap H_m(M) = o$.

Now we prove the main theorem which generalises [10, Theorem 4.3] of K. M. Benabdallah, B. J. Eisenstadt, J. M. Irwin and E. W. Poluianov.

Theorem (8.7). Let M be an S_2-module satisfying the condition (A). If some large submodule of M is decomposable then every large submodule of M is decomposable.

Proof. Let L be a large submodule of M which is decomposable. Then $Soc(L) = Soc(H_n(M))$ for some n. Now applying Theorem (3.10), we can find a basic submodule B of M with $B = \oplus \sum_{n=1}^{\infty} B_n$ where each B_n is a direct sum of uniserial modules of length n such that $M = (B_1 \oplus B_2 \oplus \ldots \oplus B_n) \oplus (B_n^* + H_n(M))$ and $B_n^* = B_{n+1} \oplus B_{n+2} \oplus \ldots$. Now, $Soc(B_n^* + H_n(M)) = Soc(H_n(M)) = Soc(L)$. Since $Soc(L) \subseteq L$ then by Theorem (4.2), Soc(L) is decomposable. Now applying Proposition (1.40), we have $Soc(L) = \bigcup_{k=1}^{\infty} N_k$ such that $N_k \cap H_{n_k}(L) = o$ for some $n_k \geq o$. Then by Proposition (8.6) $N_k \cap H_{m_k}(M) = o$ for some m_k so that $N_k \cap H_{m_k}(B_n^* + H_n(M)) = o$. Therefore, $Soc(B_n^* + H_n(M)) = \bigcup_{k=1}^{\infty} N_k$. Thus, $B_n^* + H_n(M)$ is a direct sum of uniserial modules by Corollary (1.41). Consequently, M itself is decomposable. Therefore, every submodule of M is decomposable. Hence the result follows as a particular case.

§ 9. Closure of h-dense and large submodules

In this section, we have defined the closure of a submodule and obtained a relation between h-dense submodules and closed submodules. Also a relation between large submodules and closed submodules has been obtained.

<u>Definition</u> (9.1). A submodule K of an S_2-module M is called the closure of a submodule N in M if $K/_N = (M/_N)^1$ and is denoted by $\bar{N}$ i.e. $K = \bar{N}$. N is called closed if $N = \bar{N}$.

<u>Proposition</u> (9.2). For a submodule N of an S_2-module M, $\bar{N} = \bigcap_{n=1}^{\infty} (N + H_n(M))$.

<u>Proof</u>. Let $x \in \bar{N}$ be a uniform element. If $x \in N$, then $x \in N + H_n(M)$ for all n. So that $x \in \bigcap_{n=1}^{\infty} (N + H_n(M))$. If $x \notin N$, then $\bar{x} = x + N$ is a uniform element of $\bar{N}/_N = (M/_N)^1 = \bigcap_{n=1}^{\infty} H_n(M/_N)$. So that, using lemma (5.5), we get $\bar{x} \in \bigcap_{n=1}^{\infty} ((N+H_n(M))/_N) = (\bigcap_{n=1}^{\infty} (N+H_n(M)))/_N$. Therefore, $x \in \bigcap_{n=1}^{\infty} (N + H_n(M))$ and hence, $\bar{N} \subseteq \bigcap_{n=1}^{\infty} (N + H_n(M))$. Further, let $y \in \bigcap_{n=1}^{\infty} (N + H_n(M))$ be a uniform element. If $y \in N$, then $y \in \bar{N}$. If $y \notin N$, then $\bar{y} \in (\bigcap_{n=1}^{\infty} (N+H_n(M)))/_N$ is a uniform element. Now, $(\bigcap_{n=1}^{\infty} (N+H_n(M)))/_N = \bigcap_{n=1}^{\infty} (N+H_n(M))/_N = \bigcap_{n=1}^{\infty} H_n(M/_N) = (M/_N)^1 = \bar{N}/_N$. Consequently, $\bar{y} \in \bar{N}/_N$ yields that $\bigcap_{n=1}^{\infty} (N+H_n(M)) \subseteq \bar{N}$. Thus the assertion follows.

Now, in view of the above discussion and Proposition (7.2), we have the following :

Corollary (9.3). A submodule N of an S_2-module M is h-dense in M if and only if $\bar{N} = M$.

Proposition (9.4). If L is a large submodule of an S_2-module M. Then L is a closed submodule of M.

Proof. Since L is a large submodule of M, therefore, by Proposition (8.5), $M/_L$ is direct sum of uniserial submodules. Thus, $(M/_L)^1 = o$ and hence L is closed submodule of M.

The following proposition is a generalisation of a result of M. Zubair Khan [19, Proposition 2.2].

Proposition (9.5). If L is a large submodule of an S_2-module M then L is the closure of every submodule K of L for which $L/_K$ is h-divisible.

Proof. Let $L/_K$ be an h-divisible submodule of $M/_K$, then by Proposition (5.6), $L/_K$ is the direct summand of $M/_K$ i.e. $M/_K = L/_K \oplus N/_K$. Now, $N/_K \cong M/_L$. Then by Proposition (8.5), $M/_L$ is decomposable. Thus $(N/_K)^1 = o$, so that $(M/_K)^1 = (L/_K)^1 = L/_K$. Hence L is the closure of K.

The following theorem, a generalisation of a result of

M. Zubair Khan [19, Theorem 2.1], shows a relation between a high submodule of a large submodule and a closed submodule.

<u>Theorem</u> (9.6). Let M be an S_2-module with $M^1 \neq o$, and L be a large submodule of M, then a high submodule K of L is closed in a high submodule of M.

<u>Proof</u>. By Theorem (8.4), $L^1 \neq o$. Then by Proposition (2.21), $L/_K$ is h-divisible. Therefore, by Proposition (3.6), $M/_K$ = $L/_K \oplus T/_K$. Now, we show that T is high in M. For this, we have $T \cap M^1 = T \cap L^1 = (T \cap L) \cap L^1 = K \cap L^1 = o$. To show the maximality of T with respect to this, we need only to show that for any uniform element $x \in L$ with $x \notin T$, $(T+xR) \cap L^1 \neq o$. Suppose on contrary that $(T+xR) \cap L^1 = o$, then $(K+xR) \cap L^1 = o$, which is a contradiction. Therefore, $(T+xR) \cap L^1 \neq o$. Hence T is a high submodule of M. Now, $T/_K \cong M/_L$. Hence by Proposition (8.5), $M/_L$ is decomposable and therefore, $(T/_K)^1 = o$. Thus, K is closed in T.

CHAPTER - IV

FAIR MODULES

h-neat submodules of S_2-modules, as a substructure have been studied in [23]. In our previous chapter we have also given some of the very fundamental results on h-neat submodules. From the very definition of h-neat submodule it is evident that every h-pure submodule is h-neat. Some Mathematicians like Kertesz, Szele, Fuchs have tried to classify the groups in which both concepts coincide. Recently in [13], K. Simauti has classified the abelian groups in which every neat subgroup is pure. Analogous to this, we focus our attention on the very natural and interesting question : What are those modules in which every h-neat submodule is h-pure ? We call such modules as fair module. The main purpose of this chapter is to consider this question and we have given the structure of the modules where every h-neat submodule is h-pure. In section 10, an interesting example has been constructed to show that every S_2-module does not possess this property. In section 11, some of the very fundamental and basic results on such modules have been proved. For instance, it has been shown that if M is a fair S_2-module then every h-neat submodule of M is also a fair module (Proposition 11.1). It is further proved that if an S_2-module M is either h-divisible or is a direct sum of uniserial modules of length n and n + 1, then M is a fair module (Theorem 11.4). In section 12, some characterisation for fair modules have been given. For instance, it has been shown that if M is a fair S_3-module which is not reduced then it is h-divisible (Proposition 12.1). Also, it has been proved that a reduced fair S_3-module is bounded (Proposition 12.3). It is further shown that if M is a fair S_3-module which is reduced then $M = N \oplus K$, where N is a direct sum of uniserial modules of length n for some n and K is a direct sum of uniserial modules of length n + 1 (Proposition 12.4). Finally we have proved that if M is an S_3-module. Then M is a fair module if and only if M is either h-divisible or is a direct sum of uniserial modules of length n and n+1 for some n(Theorem 12.5).

§ 10. Definition and Examples

In this section, we have introduced the concept of fair module and have given an example of fair modules. But a very natural question arises : Does there exist any module which is not fair ? An affirmative answer to this question is given.

Definition (10.1). An S_2-module M is called a fair module if every h-neat submodule of M is h-pure.

Proposition (3.6) proved by M. Zubair Khan motivates that h-divisible modules are the examples of fair modules.

The following example shows that there exists an S_3-module which is not fair.

Example (10.2). Let $M = xR \oplus yR$ be an S_3-module such that $e(x) = 3$, $e(y) = 1$, then we show that M is not fair.

Let $N = (x_1+y)R$ where $x_1 \in xR$ with $d(^{xR}/_{x_1R}) = 1$. Since $e(x) = 3$ and $d(^{xR}/_{x_1R}) = 1$ we can obtain a submodule $x_2R \subset xR$ such that $xR > x_1R > x_2R > o$ is the unique composition series of xR. Let $\mathrm{ann}(^{xR}/_{x_1R}) = P$, $\mathrm{ann}\ (^{x_1R}/_{x_2R}) = Q$, $\mathrm{ann}(x_2R) = Q'$ and $\mathrm{ann}(yR) = Q''$. Then by Proposition (1.44), we have $xP = x_1R$, $x_1Q = x_2R$, $x_2Q' = o$ and $yQ'' = o$. Now, we firstly show that N is a uniform submodule of M. Let $f: xR \longrightarrow (x_1+y)R$ be a map given as $xr \longrightarrow (x_1+y)r$. We assert that this map is well defined onto homomorphism. Suppose $xr = o$. Now $xP = x_1R$

yields that $xPQ = x_1Q = x_2R$ which further gives $xPQQ'' = x_2Q''$. If $x_2Q'' = o$ then $r \in PQQ'' \subseteq Q''$ implies that $yr = o$. Also, $x_2Q'' = o$ implies that $Q'' \subseteq Q'$ but Q'' is maximal and hence $Q'' = Q'$. Thus $r \in PQQ' \subseteq QQ'$. But $x_1Q = x_2R$ yields $x_1QQ' = o$ so that $x_1r = o$. Thus, $(x_1+y)r = o$ and the mapping is well defined. If $x_2Q'' \neq o$ then $x_2Q''Q' = o$, so $xPQQ''Q' = o$. Therefore, $r \in PQQ''Q' \subseteq Q''$ and we get $yr = o$. Also, $r \in PQQ''Q' \subseteq QQ''Q' \subseteq QQ'$ implies that $x_1r = o$. Therefore, again $(x_1+y)r = o$. Consequently, the map $xr \longrightarrow (x_1+y)r$ is well defined. It is trivial to see that it is an epimorphism. Hence $(x_1+y)R$, being the epimorphic image of a uniform module, is uniform. Now, $H_1(M) = H_1(xR) \oplus H_1(yR) = H_1(xR) \subseteq xR$. Therefore, $x_1+y \notin H_1(M)$ i.e. $H_M(x_1+y) = o$. Hence, by Lemma (6.3), $(x_1+y)R = N$ is h-neat in M.

Now, we show that N is not h-pure in M. Trivially, $H_2(N) = o$. Now for any $r' \in Q''$, $x_1r' = (x_1+y)r' \in N$. Now, either $x_1r'R = x_1R$ or $x_1r'R < x_1R$. If $x_1r'R < x_1R$, then $H_M(x_1r') = H_M(x_2) = 2$ implies that $x_1r' \in H_2(M) \cap N$ but $x_1r' \notin H_2(N)$. Thus $N \cap H_2(M) \neq H_2(N)$. If $x_1r'R = x_1R$, then $x_1Q'' = x_1R$ i.e. $x_1Q''Q = x_1Q = x_2R$ yields $x_2 = x_1r''r_1 \in N$ where $r'' \in Q''$.

So that $H_M(x_1 r'' r_1) = H_M(x_2) = 2$. Thus, in each case we find a uniform element $z \in N \cap H_2(M)$ but $z \notin H_2(N)$. Hence M is not a fair module.

§ 11. Some basic results

The main purpose of this section is to give some useful results for the further use in subsequent articles.

The following proposition is a generalisation of a result of K. Simauti [13, Proposition 1].

Proposition (11.1). Let M be an S_2-module. If M is a fair module then every h-neat submodule of M is also a fair module.

Proof. Let N be an h-neat submodule of M. If K is an h-neat submodule of N, then K is h-neat in M. Hence K is h-pure in M. Consequently, K is h-pure in N. Thus, N is a fair module.

We know that h-neat submodule and complement submodule coincide in an S_2-module (Proposition 2.17). Now in view of definitions (2.13) and (10.1), we conclude that

Remark (11.2). Every submodule of a fair S_2-module is centre of h-purity.

The following proposition immediately follows from Theorem (6.2) :

Proposition (11.3). If M is a fair S_2-module and $N \subseteq H_n(M)$ then for any complement T of N, $T \cap H_m(M)$ is h-pure in M, for all $m \leq n$.

The following theorem, is of particular interest, will be a backbone for the characterisation of fair modules done in section 12.

Theorem (11.4). Let M be an S_2-module. If M is either h-divisible or is a direct sum of uniserial modules of length n and $n+1$ for some n, possibly lacking those of length $n+1$, then M is a fair module.

Proof. Supose that N is an h-neat submodule of M. If M is h-divisible then trivially, N is h-pure in M. Hence M is a fair module. If $M = (\oplus \Sigma U_i) \oplus (\oplus \Sigma V_j)$, where each U_i is a uniserial module of length n and each V_j is a uniserial module of length $n+1$. Trivially, $N \cap H_k(M) = H_k(N)$ for $k=1$, $k \geq n+1$. Suppose that $N \cap H_m(M) = H_m(N)$ for some m, $1 \leq m < n$. Let $x \in N \cap H_{m+1}(M)$ be a uniform element. Then there exists a uniform element $y \in M$ such that $x \in yR$ and $d({}^{yR}/_{xR}) = m+1$. If $y \in N$, then $x \in H_{m+1}(N)$ and therefore, N is h-pure in M. If $y \notin N$, then we can find a uniform submodule $wR/_{xR} \subseteq yR/_{xR}$ such that $d({}^{wR}/_{xR}) = m$. Then, by assumption there exists a uniform element $t \in N$ such that $x \in tR$ and $d({}^{tR}/_{xR}) = m$. Now, appealing to (II), there exists an isomorphism $f: wR \longrightarrow tR$, defined as $wr \longrightarrow tr$, which is the identity on xR. Trivially, the map $g : wR \longrightarrow (w-f(w))R$, given as $wr \longrightarrow (w-f(w))r$, is an R-epimorphism with $xR \subseteq \ker g$. So, $e(w-f(w)) \leq d({}^{wR}/_{xR}) = m$. Let $w = w_1+w_2+\ldots+w_r+w_1'+w_2'+\ldots+w_s'$ and $t = t_1+t_2+\ldots$ $\ldots+t_r+t_1'+t_2'+\ldots+t_s'$ where $w_i, t_i \in U_i$ and $w_j', t_j' \in V_j$ for each $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,s$. Also, $(w_i-t_i)R$, being homomorphic image of $(w-t)R$, $e(w_i-t_i) \leq e(w-t) \leq m$. Similarly, $e(w_j'-t_j') \leq m$. Since, U_i and V_j are uniform submodules of length n and $n+1$ respectively, we get $H(w_i-t_i) \geq n-m$ and $H(w_j'-t_j') \geq n+1-m$. Since, $H(w-t) = \min\{H(w_i-t_i), H(w_j'-t_j')\} \geq n-m \geq 1$, therefore, $w-t \in H_1(M)$. Also, since $d({}^{yR}/_{wR}) = 1$, we get $w \in H_1(M)$ so that $t \in N \cap H_1(M) = H_1(N)$ i.e. there exists a uniform element $z \in N$ such that $t \in zR$ and $d(^{zR}/_{tR}) = 1$. Consequently, $x \in zR$ and $d(^{zR}/_{xR}) = n + 1$. Hence, $x \in H_{n+1}(N)$. Therefore, by induction, N is h-pure in M. Hence M is a fair module.

Towards the end of this section, we have the following theorem which shows that if h-neat submodules are supported by h-dense subsocles of the module then the module itself is fair. The proof immediately follows from Theorem (7.9). Hence it is omitted.

Theorem (11.5). If M is an S_2-module and every h-neat submodule of M is supported by an h-dense subsocle then M is a fair module.

§ 12. Some characterizations

It is well known that every pure subgroup of an abelian group is a neat subgroup. But in [13], K. Simauti has characterized those abelian groups in which every neat subgroup is pure. The very natural question arises : What are those S_2-modules in which every h-neat submodule is h-pure? Such modules are the central themes of this section.

Firstly, we prove the following :

Proposition (12.1). Let M be an S_3-module. If M is fair and is not reduced then M is h-divisible.

Proof. Suppose on contrary that M is not h-divisible, then by Proposition (3.4), there exists a uniform element $x \in Soc(M)$ such that $H_M(x) = k < \infty$. So we can find a uniform element $y \in M$ such that $x \in yR$ and $d({}^{yR}/_{xR}) = k$. Obviously, $H_M(y) = o$. Also by Proposition (2.4), yR is a bounded summand of M. Let T be a non-trivial h-divisible submodule of M. Then T is a summand of M. Therefore, $yR \oplus T$ is also a summand of M. Let $x_1, x_2, \ldots$ $\ldots, x_{k+2}, \ldots$ be uniform elements of T such that $e(x_1) = 1$ and $d({}^{x_{i+1}R}/_{x_iR}) = 1$ for $i = 1, 2, \ldots$. Let $yR > y_1R > \ldots > y_kR = xR > o$ be the unique composition series of yR and $ann({}^{yR}/_{y_1R}) = P_1$, $ann({}^{y_1R}/_{y_2R}) = P_2, \ldots, ann({}^{y_{k-1}R}/_{y_kR}) = P_k$, $ann(y_kR) = P$. Then by Proposition (1.44), $yP_1 = y_1R$, $y_1P_2 = y_2R, \ldots, y_{k-1}P_k = y_kR$ and $y_kP = o$. Also, let $ann({}^{x_{i+1}R}/_{x_iR}) = Q_i$ and $ann(x_1R) = Q$. Then $x_{k+2}Q_{k+1} = x_{k+1}R$, $x_{k+1}Q_k = x_kR, \ldots, x_2Q_1 = x_1R$ and $x_1Q = o$. Without any loss of generality we can assume

$Q_1, \ldots Q_{k+1}, Q, P_1, P_2, \ldots, P_k, P$ to be distinct. Now consider the submodule $N = (y + x_{k+2})R$. We show that the map $x_{k+2}R \longrightarrow (y+x_{k+2})R$ given as $x_{k+2}r \longrightarrow (y+x_{k+2})r$, is well defined. Let $x_{k+2}r = o$ for some $r \in R$. We have $x_{k+2}\, Q_{k+1}\, Q_k \ldots Q_1 P_1 = x_1 P_1$. Trivially, $x_1 P_1 \neq o$, otherwise $P_1 = Q$. Also $x_1 P_1 P_2 = o$ implies that $o = x_1 P_1 P_2 = x_1 R P_2 = x_1 P_2$, which is not possible. Thus, it is easy to see that $x_1 P_1 P_2 \ldots P_k P \neq o$. Hence $x_{k+2}\, Q_{k+1}\, Q_k \ldots Q_1 P_1 P_2 \ldots$ $\ldots P_k PQ = o$ gives that $r \in Q_{k+1}\, Q_k \ldots Q_1 P_1 P_2 \ldots P_k PQ \subseteq P_1 P_2 \ldots$ $\ldots P_k P$. So that $yr \in yP_1 P_2 \ldots P_k P = o$ i.e. $yr = o$ and therefore, the map $x_{k+2}r \longrightarrow (y+x_{k+2})r$ is well defined. Hence $y+x_{k+2}$ is a uniform element of M. Also, $H_M(y+x_{k+2}) = \min \{H_M(y), H_M(x_{k+2})\} = o$. Hence by lemma (6.3), the submodule N is h-neat in M.

Now, $H_{k+2}(N) \subseteq H_{k+2}(yR \oplus x_{k+2}R) = H_{k+2}(yR) \oplus H_{k+2}(x_{k+2}R) = o$. Therefore, $H_{k+2}(N) = o$. Also, for every $r' \in P_1 P_2 \ldots P_k P$, we have $yr' = o$ so that $x_{k+2}r' = (y+x_{k+2})r' \in N$. Since, $x_{k+2}r' \in N$, $H_M(x_{k+2}r') = \infty$. Since N is bounded, therefore, every uniform element of N will have finite height in N. Hence N is not h-pure in M, which is a contradiction as M is a fair module. Hence the result follows.

As a consequence to the above proposition we have

Corollary (12.2). A fair S_3-module is either reduced or h-divisible.

Now from the above corollary it is obvious that every reduced fair S_3-module can not be h-divisible, but it does not give the clear picture of the structure of the module. The following proposition shows that such a module will be necessarily bounded.

Proposition (12.3). A reduced fair S_3-module is bounded.

Proof. Let $B = \oplus \Sigma B_n$, where each B_n is a direct sum of uniserial modules of length n, be a basic submodule of M. Then by Theorem (3.10), $M = (B_1 \oplus B_2 \oplus \ldots \oplus B_n) \oplus \left\{B_n^* + H_n(M)\right\}$, where $B_n^* = B_{n+1} \oplus B_{n+2} \oplus \ldots$. If M is not bounded, then so is B. Hence for any integer n, we can find integers k and m such that $n \geq k \geq m + 2$ and $B_k \neq o$, $B_m \neq o$. Let $x \in B_m$ be a uniform element with $e(x) = m$, then $H(x) = o$. Also, let $y \in B_k$ be a uniform element such that $e(y) = k-1$, then $H_M(y) = 1$. Now consider the submodule $N = (x+y)R$. Let $xR > x_1R > \ldots > x_{m-1}R > o$ and $yR > y_1R > \ldots > y_{k-2}R > o$ be the unique composition series of xR and yR respectively. Then by (III), we get prime ideals

$Q_1, Q_2, \ldots, Q_{m-1}, Q_m$ and $P_1, P_2, \ldots, P_{k-2}, P_{k-1}$ such that
$xR > xQ_1 > xQ_1Q_2 > \ldots > xQ_1Q_2 \ldots Q_{m-1}Q_m = o$ and
$yR > yP_1 > yP_1P_2 > \ldots > yP_1P_2 \ldots P_{k-2}P_{k-1} = o$ are the unique composition series of xR and yR respectively. As done in Proposition (12.1), we can show that N is h-neat submodule of M.

Now, for any $r' \in Q_1Q_2 \ldots Q_{m-1}Q_m$, $xr' = o$, so that $yr' = (x+y)r' \in N$. Then either $yr'R = yR$ or $yr'R < yR$.

Case I. If $yr'R = yR$. Then $y = yr'r''$ for some $r'' \in R$ and we get $y \in N$ such that $H_N(y) \leq H(y)_{xR \oplus yR} = o$. Hence $H_N(y) \neq H_M(y)$ and therefore N is not h-pure in M.

Case II. If $yr'R < yR$. Then $H_N(yr') < e(y)-e(yr')$ while $H_M(yr') = e(y) - e(yr')+1$ and so again N is not h-pure in M. Thus, we get a contradiction. Hence the result follows.

After obtaining the boundedness of the module, we further specify the nature of the boundedness of the module in the following proposition.

Proposition (12.4). Let M be an S_3-module. If M is fair and reduced, then $M = N \oplus K$, where N is a direct sum of uniserial modules of length n for some n and K is a direct sum of uniserial modules of length $n + 1$, possibly lacking the latter submodule K.

<u>Proof</u>. By Proposition (12.3), M is bounded. Hence, by Proposition (1.42), M is a direct sum of uniserial modules. We assert that these uniserial modules are either of length n or of length $n+1$ for some n. Suppose on contrary that there exist uniform modules xR and yR of M such that $d(yR) \geq d(xR) + 2$. Then as done in Proposition (12.3), we can show that there exists a uniform element $y_1 \in yR$ such that $(x+y_1)R$ is an h-neat submodule of M which is not h-pure. Hence, M is not a fair module, a contradiction. Therefore, the assertion follows.

Now, we are in a position to give a characterisation of a fair module which generalises the main result of K. Simauti [13] and the proof follows from Propositions (11.4), (12.3), (12.4) and Corollary (12.2).

<u>Theorem</u> (12.5). Let M be an S_3-module. Then M is a

fair module if and only if M is either h-divisible or is a direct sum of uniserial modules of length n and $n+1$ for some n, possibly lacking those of length $n+1$.

CHAPTER - V

A SPECIAL STUDY IN S_2-MODULES

K. Honda [7,8] introduced the subgroups $n^{-1}H$ for any subgroup H of an abelian group G and integers $n \geq o$. With the help of this notion of subgroup he studied horizontal exponent and some decomposition theorems. In 1977, M. Zubair Khan [20] generalised these notions for S_2-modules. The main purpose of this chapter is to study S_2-modules precisely based on these notions. If M is an S_2-module then $H^k(N)$, for any submodule N of M and integer $k \geq o$, denotes the submodule generated by those uniform elements $x \in M$ for which there is atleast a uniform element $y \in xR \cap N$ such that $d(^{xR}/_{yR}) \leq k$. In first section, we have proved some basic results some of which are quite interesting. For instance, it is proved that for any submodule N and K of M, $H_n(H^n(N)) = N \cap H_n(M)$ and $H_m(N \cap H^{m+n}(K)) = H_m(N) \cap H^n(K)$. In the second section, we have discussed horizontal exponent and deduced a number of results on it. The last section deals with some applications of this notion for h-purity and it is proved that for any submodule N of an S_2-module M , the following are equivalent :

(a) N is h-pure in M,

(b) N is a direct summand of $H^n(N)$ for every $n \geq o$,

(c) If $N \subseteq K \subseteq M$ and $K/_N$ is finitely generated then N is a direct summand of K.

Though this chapter does not involve the notions of previous chapters, but it has its own importance and implications.

§ 13. Some basic results

In this section, we firstly present a new concept which generalises the corresponding concept of K. Honda [7,8] and then a number of basic results based on this concept are proved.

Definition (13.1). Let M be an S_2-module and N be a submodule of M then for any integer $k \geq o$, we define $H^k(N)$ to be submodule of M generated by those uniform elements $x \in M$, for which the elements $\bar{x} = x + N$ in $M/_N$ has exponent $\leq k$.

In other words $H^k(N)$ is the submodule generated by those uniform elements x for which $d(^{xR}/_{xR \cap N}) \leq k$ i.e. there exists atleast a uniform element $y \in xR \cap N$ such that $d(^{xR}/_{yR}) \leq k$.

K. Honda introduced the subgroups $n^{-1}H$ for any subgroup H of a group G and integer $n \geq o$. It is trivial to see that the submodules $H^k(N)$ are the generalised form of the subgroups $k^{-1}N$.

Now we shall prove some of the basic results which generalise the results of K. Honda [7,8]. Whenever the proof of a result runs analogous to that of K. Honda's result, the proofs are either omitted are sketchy.

Firstly we prove the following result :

<u>Proposition</u> (13.2). Let M be an S_2-module and N,K be submodules of M then the following hold :

(a) $N \subseteq H^n(N)$ for $n \geq o$

(b) $Soc(M) = H^1(o)$

(c) $Soc(M) \subseteq H^n(N)$ for $n \geq 1$

(d) If $N \subseteq K$ then $H^n(N) \subseteq H^n(K)$ for $n \geq o$

(e) $H^n(N) \subseteq H^k(N)$ for $n \leq k$

(f) $H^n(H^k(N)) = H^{n+k}(N)$

(g) $H^n(N)$ is essential submodule of M for $n \geq 1$

(h) $H_n(H^n(N)) = N \cap H_n(M)$ for $n \geq o$

(i) $H_n(N) \subseteq K$ if and only if $N \subseteq H^n(K)$

(j) $H^n(N \cap K) = H^n(N) \cap H^n(K)$.

Proof. (a), (b), (c), (d), (e) and (f) trivially follow from the definition. Since $Soc(M)$ is essential, therefore (g) follows from (c).

(h) Let x be a uniform element in $N \cap H_n(M)$. Now we obtain a uniform element $y \in M$ such that $x \in yR$ and $d({}^{yR}/_{xR})=n$. Since $x \in N$, so $y \in H^n(N)$. Therefore, $x \in H_n(H^n(N))$ and we get $N \cap H_n(M) \subseteq H_n(H^n(N))$. For the other inclusion let z be a uniform element in $H_n(H^n(N))$ then there exists a uniform element $x \in H^n(N)$ such that $d({}^{xR}/_{zR}) = n$. Trivially $z \in H_n(M)$. As $x \in H^n(N)$, we can obtain a uniform element $y \in xR \cap N$ such that $d({}^{xR}/_{yR}) \leq n$. Since xR is totally ordered, either $zR \subseteq yR$ or $yR \subseteq zR$. As $d({}^{xR}/_{zR}) = n$, we must have $zR \subseteq yR$. Hence $z \in N$ and we got $z \in N \cap H_n(M)$ which further yields $H_n(H^n(N)) \subseteq N \cap H_n(M)$. Therefore, $H_n(H^n(N)) = N \cap H_n(M)$.

(i) Let $H_n(N) \subseteq K$. Consider a uniform element $x \in N$

and suppose $x \notin H^n(K)$. In other words $d({}^{xR}/_{xR \cap K}) > n$. Let $d({}^{xR}/_{xR \cap K}) = n + m$, then we can find a uniform element $y \in xR$ such that $d({}^{yR}/_{xR \cap K}) = m$. Then $d({}^{xR}/_{yR}) = n$ and hence $y \in H_n(N)$. But $H_n(N) \subseteq K$, therefore $y \in K$, consequently $yR \subseteq xR \cap K$. But $xR \cap K \subseteq yR$, so $yR = xR \cap K$ which is not possible as $d({}^{yR}/_{xR \cap K}) = m$. Therefore, $x \in H^n(K)$ and we get $N \subseteq H^n(K)$. Conversely, suppose $N \subseteq H^n(K)$ then $H_n(N) \subseteq H_n(H^n(K))$. Hence appealing to (h) we get $H_n(N) \subseteq K \cap H_n(M)$. Therefore, $H_n(N) \subseteq K$.

(j) Obviously left hand side is contained in the right hand side. Consider a uniform element $x \in H^n(N) \cap H^n(K)$. By definition $d({}^{xR}/_{xR \cap N}) \leq n$, $d({}^{xR}/_{xR \cap K}) \leq n$. As xR is uniserial, $xR \cap K \cap N = xR \cap N$ or $xR \cap K$. Consequently $x \in H^n(N \cap K)$. This proves (j).

<u>Proposition</u> (13.3). For any submodule N of M, $H^n(H_n(N)) = N + H^n(o)$.

<u>Proof</u>. Consider a uniform element $x \in H^n(H_n(N))$. If $e(x) \leq n$, $x \in H^n(o)$. Let $e(x) > n$. By definition $yR = xR \cap H_n(N) \neq o$ and $d({}^{xR}/_{yR}) \leq n$. Now $y \in H_n(N)$ implies that there exists a

uniform element $z \in N$ such that $y \in zR$ and $d(^{zR}/_{yR}) = n$. Condition(II) on M gives a monomorphism $f : xR \longrightarrow zR$, which is identity on yR. Then $x - f(x)$ is a uniform element with $e(x - f(x)) \leq n$. Hence $x = f(x) + (x - f(x)) \in N + H^n(o)$. This gives $H^n(H_n(N)) \subseteq N + H^n(o)$. Also $N + H^n(o) \subseteq H^n(H_n(N))$ follows from Proposition (13.2) (i). Hence $H^n(H_n(N)) = N + H^n(o)$.

Proposition (13.4). For any two submodules N and K of M, $H_m(N \cap H^{m+n}(K)) = H_m(N) \cap H^n(K)$.

Proof. Let x be a uniform element in $H_m(N) \cap H^n(K)$. This implies that there exists a uniform element $z \in N$ such that $x \in zR$ and $d(^{zR}/_{xR}) = m$. As also $x \in H^n(K)$, we get $z \in H^{n+m}(K)$. Consequently, $z \in H^{n+m}(K) \cap N$. As $d(^{zR}/_{xR}) = m$, therefore, $x \in H_m(N \cap H^{m+n}(K))$. Hence $H_m(N) \cap H^n(K) \subseteq H_m(N \cap H^{m+n}(K$ For proving the reverse inclusion, consider a uniform element $y \in H_m(N \cap H^{m+n}(K))$, then there exists a uniform element $u \in N \cap H^{m+n}(K)$ such that $y \in uR$ and $d(^{uR}/_{yR}) = m$. Then obviously $y \in H_m(N)$. Appealing to Proposition (13.2) (f) we get $H^{m+n}(K) = H^m(H^n(K))$. Hence $u \in H^m(H^n(K))$. Therefore, appealing to the definition we get $y \in H^n(K)$. Hence

$H_m(N \cap H^{m+n}(K)) \subseteq H_m(N) \cap H^n(K)$. This proves the Proposition.

§ 14. Horizontal exponent

In this section, we define horizontal exponent of a module and deduce a number of results. Since the proofs of the most of the results run parallel to that of K. Honda [7,8]. Therefore, they are omitted.

Definition (14.1). An S_2-module M with $M \neq H_1(M)$, is said to be of horizontal exponent n if $H^{n-1}(o) \subseteq H_1(M)$ but $H^n(o) \not\subseteq H_1(M)$, symbolically we write $h(M) = n$.

Definition (14.2). An S_2-module M is said to be an elementary module of exponent n if $h(M) = n$ and $H^n(o) = M$.

Proposition (14.3). If M is an elementary module of exponent $n \geq 2$, and if $H_1(M) = \oplus \Sigma a_i R$, where a_iR are uniserial submodules, each of same length $n-1$. Then $M = \oplus \Sigma c_iR$ such that c_iR are uniserial modules with $a_i \in c_iR$ and $d({}^{c_iR}/{}_{a_iR}) = 1$.

Proof. Consider any uniform element $x \in M' = \oplus \Sigma c_iR$ then we have $x = b_{i_1} + b_{i_2} + \dots + b_{i_t}$ for some $b_{i_j} \in c_{i_j}R$.

Then $e(x) = \max\{e(b_{i_j})\}$. Hence, $H_{M'}(x) \geq n - e(x)$. However by hypothesis M has no uniform element of exponent exceeding n. Consequently $H_{M'}(x) \leq n - e(x)$. Hence $H_M(x) = H_{M'}(x)$. This proves that M' is an h-pure submodule of M. Hence by Proposition (2.3), M' is a summand of M. This gives $M = M'$ as M' is essential submodule of M.

Applying the same technique as above and proceeding on similar lines as in [7, Proposition 9], we have the following :

<u>Proposition</u> (14.4). If M is an S_2-module and K is submodule of M. If M and K are both elementary modules of exponent n $(n \geq 2)$ and if $H^{n-1}(o) = H_1(M) = N \oplus H_1(K)$ where N is an elementary module of exponent n-1, then there exists an elementary submodule N' of exponent n such that $M = N' \oplus K$, $N' \supseteq N$.

The proof of the following result is given only to point out the technique how the proofs of the other results can be adopted from K. Honda [7,8].

<u>Proposition</u> (14.5). If $h(M) = n \geq 2$ is finite and N is a submodule of M such that $H_1(M) \subseteq N$ and $H^n(o) \nsubseteq N$ then $h(N) = n - 1$.

Proof. Appealing to the definition and assumption we get $H^{n-1}(o) \subseteq H_1(M) \subseteq N$. Hence $H_1(H^{n-1}(o)) \subseteq H_1(N)$. Since $n \geq 2$, using Proposition (13.2) (h), we get $H_1(H^{n-1}(o)) = H_1(H^1(H^{n-2}(o))) = H^{n-2}(o) \cap H_1(M)$. Since $H^{n-2}(o) \subseteq H^{n-1}(o)$, we have $H^{n-2}(o) \cap H_1(M) = H^{n-2}(o) = H^{n-2}(o) \cap N$. Therefore, $H^{n-2}(o) \cap N \subseteq H_1(N)$. Now to complete the proof we shall show that $H^{n-1}(o) \cap N \not\subseteq H_1(N)$. Suppose on contrary $H^{n-1}(o) \cap N \subseteq H_1(N)$. Since $H^{n-1}(o) \subseteq N$, $H^{n-1}(o) \cap N = H^{n-1}(o)$. Now $H^1(H_1(N)) \supseteq H^1(H^{n-1}(o))$ $H^n(o)$. Appealing to Proposition (13.3), we get $H^1(H_1(N)) = N + H^1(o) \supseteq H^n(o)$. Since $n \geq 2$, $H^1(o) \subseteq H^{n-1}(o) \subseteq N$. Hence $N + H^1(o) = N$ and we get $H^n(o) \subset N$ which is a contradiction. Therefore, $H^{n-1}(o) \cap N \not\subseteq H_1(N)$ and we get $h(N) = n-1$.

Corollary (14.6). If $h(M) = n \geq 2$ is finite then $h(H_1(M)) = n - 1$.

Proof. Taking $N = H_1(M)$ in Proposition (14.5), the result follows.

Proposition (14.7). If $h(M) \geq n$ then $H^n(o)$ is an elementary module of exponent n and $H_1(H^n(o)) = H^{n-1}(o)$.

Proof. It is similar as in [7, Proposition 6].

Proposition (14.8). If $h(M) = n$ then the following hold :

(a) $Soc(M) \subseteq H_{n-1}(M)$.

(b) $Soc(M) \nsubseteq H_n(M)$.

(c) $M/H_n(M)$ is an elementary module of exponent n.

Proof. (a) and (b) are immediate from the definition of $h(M)$. The proof of (c) is analogous to [8, Lemma 4].

Now we generalise [7, Decomposition Theorem]. Since the proof runs on similar lines, it is omitted.

Theorem (14.9). If M is an S_2-module and $h(M) = n (1 \leq n \neq \infty)$ then $M = K_1 \oplus K_2$ where K_1 is an elementary module of exponent n and either $K_2 = o$ or $h(K_2) > n$.

§ 15. Applications

In this section, we use the concept of the submodules $H^k(N)$ and deduce a number of useful results of h-pure submodules.

Proposition (15.1). A submodule N of an S_2-module M is h-pure if and only if $H_n(H^n(N)) = H_n(N)$ for all $n \geq o$.

Proof. By Proposition (13.2) (h), we have $H_n(H^n(N)) = N \cap H_n(M)$, therefore, using the definition of h-purity we get N to be h-pure if and only if $H_n(H^n(N)) = H_n(N)$.

Now we give another criterion for h-purity of a submodule.

<u>Proposition</u> (15.2). If M is an S_2-module, then a submodule N of M is h-pure if and only if $H^n(N) = N + H^n(o)$.

<u>Proof</u>. For the necessity let x be a uniform element in $H^n(N)$ such that $x \notin N$ then $e(\bar{x}) \leq n$, $\bar{x} \in H^n(N)/_N$. Let $e(\bar{x}) = t$. By Theorem (4.12), there exists a uniform element $y' \in M$ such that $e(y') = e(\bar{x}) = t$ and $\bar{y}' = \bar{x}$. This gives $x - y' \in N$ and $x \in N + H^n(o)$ as $y' \in H^t(o) \subseteq H^n(o)$. Therefore, $H^n(N) \subseteq N + H^n(o)$. The other inclusion is trivial. Therefore, $H^n(N) = N + H^n(o)$. Conversely, $H_n(H^n(N)) = H_n(N) + H_n(H^n(o)) = H_n(N)$. So by Proposition (15.1), N is an h-pure submodule of M.

<u>Proposition</u> (15.3). If N and K are h-pure submodules of M then $H^n(N) \subseteq H^n(K)$ if and only if $H_n(N) \subseteq H_n(K)$.

<u>Proof</u>. Suppose $H_n(N) \subseteq H_n(K)$ then $H^n(H_n(N)) \subseteq H^n(H_n(K))$. Appealing to Proposition (13.3), we get $N + H^n(o) \subseteq K + H^n(o)$. Hence by Proposition (15.2), we get $H^n(N) \subseteq H^n(K)$. Conversely,

suppose $H^n(N) \subseteq H^n(K)$ then $H_n(H^n(N)) \subseteq H_n(H^n(K))$. Appealing to Preposition (15.1), we get $H_n(N) \subseteq H_n(K)$.

Now in view of the results proved we have the following which generalizes [15, Theorem 28.4], for torsion abelian groups. Since the proof is similar, it is emmitted.

<u>Theorem</u> (15.4). Let N be a submodule of an S_2-module M then the following are equivalent :

(a) N is h-pure in M,

(b) N is a direct summand of $H^n(N)$ for every $n \geq o$,

(c) If $N \subseteq K \subseteq M$ and $K/_N$ is finitely generated then N is a direct summand of K.

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